Cg-Closed Sets And C-Normal Spaces

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Abstract: In this paper, we introduced a new class of sets called C-generalized closed (briefly Cg-closed) set which is a simultaneous generalization of C-closed and g-closed sets. First we investigated basic some properties of Cg-closed sets and then we obtained the relationship of Cg-closed sets with some other existing generalized closed sets. Moreover, we introduced the notion of C-Normal space by using C-closed sets, also we obtained some basic characterizations, properties and preservation theorems of C-normal spaces. Further, we also introduced some function related to Cg-open sets and investigated their properties with C-normal spaces.

Keyword: C-closed set, F-closed sets, Cg-closed set, Fg-closed set, C-normal space, almost Cg-closed function, almost Cg-continuous function etc.

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1. Introduction

2. Preliminaries

Throughout in this paper, spaces \((X, \mathcal{I})\), \((Y, \sigma)\), and \((Z, \gamma)\) (or simply \(X\), \(Y\) and \(Z\)) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let \(f: X \rightarrow Y\) (or simply \(f\)) always denote a mapping from space \(X\) to space \(Y\). Let \(B\) be a subset of a space \(X\). The closure of \(B\), interior of \(B\) and complement of \(B\) is denoted by \(\text{cl}(B)\), \(\text{int}(B)\) and \(B^c\) (or \(X - B\)) respectively.

**Definition 2.1**: A subset \(B\) of a topological space \((X, \mathcal{I})\) is said to be:

1. **regular open** [7] if \(B = \text{int} (\text{cl}(B))\).
2. **semi open** [4] if \(B \subseteq \text{cl}(\text{int}(B))\).
3. **F-open** [1] if \(\text{cl}(B) - B\) is finite set and \(B\) is open in \(X\).
4. **C-open** [2] if \(\text{cl}(B) - B\) is countable set and \(B\) is open in \(X\).

The complement of a regular open (resp. semi open, F-open and C-open set) set is called **regular closed** (resp. semi closed, F-closed and C-closed) set.

The intersection of all regular closed (resp. semi closed, F-closed and C-closed) sets containing \(B\), is called **regular closure** (resp. semi closure, F-closure and C-closure) of \(B\), and is denoted by \(\text{r-cl}(B)\) (resp. \(\text{s-cl}(B)\), \(\text{F-cl}(B)\) and \(\text{C-cl}(B)\)). The union of all regular open (resp. semi open, F-open and C-open) sets contained in \(B\), is called **regular interior** (resp. semi interior, F-interior and C-interior) of \(B\), and is denoted by \(\text{r-int}(B)\) (resp. \(\text{s-int}(B)\), \(\text{F-int}(B)\) and \(\text{C-int}(B)\)).

The collection of all regular open (resp. semi open, F-open and C-open) sets in \(X\) is denoted by \(\text{r-O}(X)\) (resp. \(\text{s-O}(X)\), \(\text{F-O}(X)\) and \(\text{C-O}(X)\)). The collection of all regular closed (resp. semi closed, F-closed and C-closed) sets in \(X\) is denoted by \(\text{r-C}(X)\) (resp. \(\text{s-C}(X)\), \(\text{F-C}(X)\) and \(\text{C-C}(X)\)).

**Remark 2.2** From the above definitions the relationship among C-open sets and some other existing weaker and stronger forms of open sets are given as:

F-open \(\rightarrow\) C-open \(\rightarrow\) open \(\rightarrow\) semi open

Where none of the implications is reversible can be seen from the following examples:

**Example 2.3** Let \(X = \{a, b, c\}\) and \(\mathcal{I} = \{\phi, \{a\}, X\}\). Then \(\{a, b\}\) is semi open set in \(X\) but not open set in \(X\).

**Example 2.4** Let \((\mathbb{R}, \mathcal{U})\) be the usual topological space then interval \([2, 5)\) is semi open in \(\mathbb{R}\) as \([2, 5) \subset \text{cl} (\text{int}([2, 5]))\) but not open in \(\mathbb{R}\).

**Example 2.5** Let \(X = \mathbb{R}\) and \(\mathcal{I}\) is the collection of all those subsets of \(\mathbb{R}\) which do not contain any irrational numbers together with \(\mathbb{R}\) then \((\mathbb{R}, \mathcal{I})\) be a topological space. Now the set of rational number \(\mathbb{Q}\) be an open set in \((\mathbb{R}, \mathcal{I})\) but not a C-open set in \((\mathbb{R}, \mathcal{I})\) as: \(\text{cl}(\mathbb{Q}) - \mathbb{Q} = \mathbb{R} - \mathbb{Q} = \mathbb{Q}^c\) (set of irrational numbers) which is an uncountable set.

**Example 2.6** Let \(X = \mathbb{R}\) and \(\mathcal{I}\) is the collection of all those subsets of \(\mathbb{R}\) which contains a particular point 0 together with empty set \(\phi\) then \((\mathbb{R}, \mathcal{I})\) be a topological space. Now the set of integer \(\mathbb{Z}\) be an open set in \((\mathbb{R}, \mathcal{I})\) but not C-open set in \((\mathbb{R}, \mathcal{I})\) as: \(\text{cl}(\mathbb{Z}) - \mathbb{Z} = \mathbb{R} - \mathbb{Z}\) which is not a countable set.

**Example 2.7** The set of natural numbers \(\mathbb{N}\) is a closed set of usual topological spaces \((\mathbb{R}, \mathcal{U})\) then \(\mathbb{R} - \mathbb{N}\) is open set in \(\mathbb{R}\), also C-open set in \(\mathbb{R}\) but not F-open set in \(\mathbb{R}\) as: \(\text{cl}(\mathbb{R} - \mathbb{N}) - (\mathbb{R} - \mathbb{N}) = \mathbb{R} - (\mathbb{R} - \mathbb{N}) = \mathbb{N}\) which is countable set but not finite set.
Definition 2.8 A subset B of a topological space \((X, \mathcal{J})\) is said to be:

1. **g-closed** [5] if \(\text{cl}(B) \subseteq U\) whenever \(A \subseteq U\) and \(U \in \mathcal{J}\).
2. **s*g-closed** [6] if \(\text{cl}(B) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open.
3. **Fg-closed** [3] if \(\text{cl}(B) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is F-open.

3. Cg-closed sets

Definition 3.1 A subset B of a topological space \((X, \mathcal{J})\) is said to be **Cg-closed** if \(\text{cl}(B) \subseteq U\) whenever \(B \subseteq U\) and \(U\) is C-open. The complement of the Cg-closed set is called Cg-open set. The collection of all Cg-open (resp. Cg-closed) sets is denoted by \(\text{Cg-O}(X)\) (resp. \(\text{Cg-C}(X)\)).

The intersection of all Cg-closed sets containing B, is called the Cg-closure of B and is denoted by \(\text{Cg-cl}(B)\). The Cg-interior of B, denoted by \(\text{Cg-int}(B)\) is defined to be the union of all Cg-open sets contained in B.

Theorem 3.2 Every s*g-closed set is Cg-closed set.

**Proof:** Let \(B\) be an s*g-closed set in \(X\) and let \(B \subseteq U\) where \(U\) is C-open in \(X\). Now every C-open set is semi open and \(B\) is s*g-closed, so by the definition of s*g-closed set, \(\text{cl}(B) \subseteq U\), hence \(B\) is Cg-closed set in \(X\).

Theorem 3.3 Every g-closed set is Cg-closed set.

**Proof:** Let \(B\) be a g-closed set in \(X\) and let \(B \subseteq U\) where \(U\) is C-open in \(X\). Now every C-open set is open set and \(B\) is g-closed, so by the definition of g-closed set, \(\text{cl}(B) \subseteq U\), hence \(B\) is Cg-closed set in \(X\).

Theorem 3.4 Every Cg-closed set is Fg-closed set.

**Proof:** Let \(B\) be a Cg-closed set in \(X\) and let \(B \subseteq U\) where \(U\) is F-open in \(X\). Now every F-open set is C-open set and \(B\) is Cg-closed, so by the definition of Cg-closed set, \(\text{cl}(B) \subseteq U\), hence it is clear that \(B\) is Fg-closed set in \(X\).

Remark 3.5 We summarize the fundamental relationships between several types of generalized closed sets by the following implications:

\[
\text{closed} \rightarrow \text{s*g-closed} \rightarrow \text{g-closed} \rightarrow \text{Cg-closed} \rightarrow \text{Fg-closed}
\]

The converse of the above implication may not be true as can be seen from the following examples:

**Example 3.6** Let for the set of real numbers \(\mathbb{R}\), the collection of open sets \(\mathcal{J} = \{\emptyset, \mathbb{Q}, \mathbb{Q}^c, \mathbb{R}\}\) then \((\mathbb{R}, \mathcal{J})\) be a topological space. The set of integer \(\mathbb{Z}\) is not closed in \((\mathbb{R}, \mathcal{J})\) as \(\text{cl}(\mathbb{Z}) = \mathbb{Q}\), but \(\mathbb{Z}\) is an s*g-closed set as \(\mathbb{Q}\) is the smallest semi open set which contains \(\mathbb{Z}\) and \(\text{cl}(\mathbb{Z}) = \mathbb{Q} \subseteq \mathbb{Q}\).

**Example 3.7** For the set of real numbers \(\mathbb{R}\), let the collection of open sets \(\mathcal{J} = \{\emptyset, \mathbb{N}, \mathbb{R}\}\) (where \(\mathbb{N}\) is the set of natural number) then \((\mathbb{R}, \mathcal{J})\) be a topological space. Now the set of integer \(\mathbb{Z}\) is a g-closed in \((\mathbb{R}, \mathcal{J})\) as: \(\mathbb{R}\) is the smallest open set which contains \(\mathbb{Z}\) (because \(\mathbb{Z}\) is not open) and \(\text{cl}(\mathbb{Z}) = \mathbb{R}\) also contained in \(\mathbb{R}\). But \(\mathbb{Z}\) is not s*g-closed set in \((\mathbb{R}, \mathcal{J})\) as: \(\mathbb{Z}\) be a semi open in \((\mathbb{R}, \mathcal{J})\) (because \(\mathbb{Z} \subseteq \text{cl}(\text{int}(\mathbb{Z})) = \text{cl}(\mathbb{N}) = \mathbb{R}\) and \(\mathbb{Z} \subseteq \mathbb{Z}\) but \(\text{cl}(\mathbb{Z}) = \mathbb{R}\) is not subset of \(\mathbb{Z}\).

**Example 3.8** By example 2.5 the set of rational numbers \(\mathbb{Q}\) is a Cg-closed set in \((\mathbb{R}, \mathcal{J})\) as set of real numbers \(\mathbb{R}\) is the smallest C-open set containing \(\mathbb{Q}\) (because \(\mathbb{Q}\) is not C-open set in \((\mathbb{R}, \mathcal{J})\)) and \(\text{cl}(\mathbb{Q}) = \mathbb{R} \subseteq \mathbb{R}\). But \(\mathbb{Q}\) is not a g-closed set in \((\mathbb{R}, \mathcal{J})\) as \(\mathbb{Q}\) is open in \((\mathbb{R}, \mathcal{J})\) also \(\mathbb{Q} \subseteq \mathbb{Q}\) but \(\text{cl}(\mathbb{Q}) = \mathbb{R}\) is not a subset of \(\mathbb{Q}\).
Example 3.9 For topological spaces \((\mathbb{R}, \mathcal{S})\), where \(\mathcal{S} = \{\emptyset, \mathbb{N}, \mathbb{R}\}\). Now the set of natural numbers \(\mathbb{N}\) is a Cg-closed set in \((\mathbb{R}, \mathcal{S})\) as the set of real numbers \(\mathbb{R}\) is the smallest C-open set containing \(\mathbb{N}\) (because \(\mathbb{N}\) is not C-open set in \((\mathbb{R}, \mathcal{S})\) as \(\mathbb{N}\) is open and \(cl(\mathbb{N}) = \mathbb{R} - \mathbb{N}\) which is an uncountable set) and \(cl(\mathbb{N}) = \mathbb{R} \subset \mathbb{R}\). But \(\mathbb{N}\) is not a g-closed set in \((\mathbb{R}, \mathcal{S})\) as \(\mathbb{N}\) is an open set in \((\mathbb{R}, \mathcal{S})\) also \(\mathbb{N} \subset \mathbb{N}\) but \(cl(\mathbb{N}) = \mathbb{R} \subset \mathbb{R}\).

Example 3.10 For the topological space \((\mathbb{R}, \mathcal{S})\) where \(\mathbb{R}\) is the set of real numbers and \(\mathcal{S}\) be the collection of open sets and \(\mathcal{S} = \{\emptyset, \mathbb{Q}^c, \mathbb{R}\}\), the set of irrational number \(\mathbb{Q}^c\) is a C-open set in \((\mathbb{R}, \mathcal{S})\) as \(\mathbb{Q}^c\) is open set in \((\mathbb{R}, \mathcal{S})\) and \(cl(\mathbb{Q}^c) = \mathbb{R} - \mathbb{Q}^c = \mathbb{Q}\) which is a countable set. Now \(\mathbb{Q}^c\) is not a Cg-closed set in \((\mathbb{R}, \mathcal{S})\) as \(\mathbb{Q}^c\) is C-open set and \(\mathbb{Q}^c \subset \mathbb{Q}\) but \(cl(\mathbb{Q}^c) = \mathbb{R}\) is not a subset of \(\mathbb{Q}^c\), but \(\mathbb{Q}^c\) is an Fg-closed set in \((\mathbb{R}, \mathcal{S})\) as \(\mathbb{R}\) is the smallest F-open set which contains \(\mathbb{Q}^c\) (because \(\mathbb{Q}^c\) is not F-open in \((\mathbb{R}, \mathcal{S})\)) and \(cl(\mathbb{Q}^c) = \mathbb{R} \subset \mathbb{R}\).

4. Properties of Cg-closed sets

**Theorem 4.1:** Union of two Cg-closed set is Cg-closed set.

**Proof:** Let \(J\) and \(K\) be two Cg-closed sets. Let \(U\) be a C-open set containing \(J \cup K\). Now \(J\) is Cg-closed set then \(cl(J) \subset U\) as \(J \subset U\) and \(U\) is C-open set, also \(K\) is Cg-closed set then \(cl(K) \subset U\) as \(K \subset U\) and \(U\) is C-open set. Now \(cl(J) \subset U\) and \(cl(K) \subset U\) \(\Rightarrow cl(J) \cup cl(K) \subset U\) \(\Rightarrow cl(J \cup K) \subset U\) (because \(cl(J \cup K) = cl(J) \cup cl(K)\)). Hence \(cl(J \cup K) \subset U\) whenever \(J \cup K \subset U\) and \(U\) is C-open set. Hence \(J \cup K\) is Cg-closed set.

In general finite union of Cg-closed sets is Cg-closed set.

**Theorem 4.2:** Intersection of two Cg-closed set is Cg-closed set.

**Proof:** Let \(J\) and \(K\) be two Cg-closed set. Now \(J\) is Cg-closed set if \(cl(J) \subset U\) whenever \(J \subset U\) and \(U\) is C-open set, also \(K\) is Cg-closed set if \(cl(K) \subset U\) whenever \(K \subset U\) and \(U\) is C-open set. Now \(U_1 \cap U_2\) is C-open set as \(U_1\) and \(U_2\) are C-open sets, and \(J \cap K \subset U_1 \cap U_2\) as \(J \subset U_1\) and \(K \subset U_2\). Now \(cl(J) \subset U_1\) and \(cl(K) \subset U_2\) \(\Rightarrow cl(J \cap K) \subset U_1 \cap U_2\) \(\Rightarrow cl(J \cap K) \subset U_1 \cap U_2\) (because \(cl(J \cap K) = cl(J) \cap cl(K)\)). Hence \(cl(J \cap K) \subset U_1 \cap U_2\) whenever \(J \cap K \subset U_1 \cap U_2\) and \(U_1 \cap U_2\) is C-open set. Hence \(J \cap K\) is Cg-closed set.

In general finite intersection of Cg-closed sets is Cg-closed set.

**Theorem 4.3:** Union of two Cg-open sets is Cg-open set.

**Proof:** Let \(G\) and \(H\) be two Cg-open subset of a topological space \((X, \mathcal{S})\). Then \(X - G\) and \(X - H\) be two closed Cg-subset of \(X\). Hence \((X - G) \cap (X - H)\) is Cg-closed subset of \(X\) by **Theorem 4.2** Now \((X - G) \cap (X - H) = X - (G \cup H)\) be Cg-closed set \(\Rightarrow G \cup H\) is Cg-open set. Hence union of two Cg-open sets is Cg-open set.

In general finite union of Cg-open sets is Cg-open set.

**Theorem 4.4:** Intersection of two Cg-open sets is Cg-open set.

**Proof:** Let \(G\) and \(H\) be two Cg-open subset of a topological space \((X, \mathcal{S})\). Then \(X - G\) and \(X - H\) be two Cg-closed subsets of \(X\). Hence \((X - G) \cap (X - H)\) be the Cg-closed subset of \(X\) by **Theorem 4.1**. Now \((X - G) \cap (X - H) = X - (G \cap H)\) be Cg-closed set \(\Rightarrow G \cap H\) is Cg-open set. Hence intersection of two Cg-open sets is Cg-open set.

In general finite intersection of Cg-open sets is Cg-open set.
Remark 4.5: Arbitrary union of $C^g$-closed sets is may not be $C^g$-closed set.

Example 4.6: For the set of natural number $\mathbb{N}$, $\exists$ be the collection of all those subset of $\mathbb{N}$ whose complement is finite together with the empty set, then $\exists$ is cofinite topology for $\mathbb{N}$. Let $A_n = \{n+1\}$ $\forall$ $n \in \{1, 2, 3, 4, \ldots\}$ be the closed sets, hence $C^g$-closed subsets in the $\mathbb{N}$. Now let $A$ be the countable union of $A_n$, i.e. $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \ldots$ = $\{2, 3, 4, 5, \ldots\}$ = $\mathbb{N} - \{1\}$ which is not $C^g$-closed set as $A \subset \mathbb{N} - \{1\}$ and $\mathbb{N} - \{1\}$ is $C$-open set (as $\mathbb{N} - \{1\}$ is open set in $\mathbb{N}$, and $cl(\mathbb{N} - \{1\}) - (\mathbb{N} - \{1\}) = \mathbb{N} - (\mathbb{N} - \{1\}) = \text{singleton set \{1\}}$ which is countable) but $cl(A) = \mathbb{N}$ which is not subset of $\mathbb{N} - \{1\}$. Hence arbitrary union of $C^g$-closed sets is may not be $C^g$-closed set.

Remark 4.7: Arbitrary intersection of $C^g$-open sets is may not be $C^g$-open set.

Example 4.8: By example 4.6, $B_n = \mathbb{N} - \{n+1\}$ $\forall$ $n \in \mathbb{N}$ be the open set, hence $C^g$-open sets in $\mathbb{N}$. Now let $B$ be the countable intersection of $B_n$, i.e. $B = B_1 \cap B_2 \cap B_3 \cap B_4 \ldots$ = $(\mathbb{N} - \{2\}) \cap (\mathbb{N} - \{3\}) \cap (\mathbb{N} - \{4\}) \cap (\mathbb{N} - \{5\}) \ldots$ = $\mathbb{N} - \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \ldots$ = $\mathbb{N} - \{2, 3, 4, 5, \ldots\}$ = $\{1\}$ which is not a $C^g$-open set as $\mathbb{N} - \{1\}$ is not $C^g$-closed set by Example 4.6. Hence arbitrary intersection of $C^g$-open sets is may not be $C^g$-open set.

Definition 4.9: The intersection of all $C$-open subsets of a space $X$ containing a set $B$ is called the $C$-kernel of $B$ and is denoted by $C$-ker($B$).

Lemma 4.10: A subset $B$ of a space $X$ is $C^g$-closed iff $cl(B) \subset C$-ker($B$).

Proof: Let $B$ be a $C^g$-closed set in $X$. Then $cl(B) \subset U$ whenever $B \subset U$ and $U$ is $C$-open in $X$. This implies $cl(B) \subset \cap \{U: B \subset U \text{ and } U \text{ is } C \text{-open in } X\}$ i.e. $cl(B) \subset C$-ker($B$).

Conversely, let $cl(B) \subset C$-ker($B$). This implies $cl(B) \subset \cap \{U: B \subset U \text{ and } U \text{ is } C \text{-open in } X\}$ i.e. $cl(B) \subset U$ whenever $B \subset U$ and $U$ is $C$-open in $X$. This proves that $B$ is $C^g$-closed.

5. C-NORMAL SPACES

Definition 5.1: A space $X$ is said to be $C$-normal (resp. normal [8]) if for every pair of disjoint $C$-closed (resp. closed) sets $J$ and $K$ in $X$, there exist disjoint open sets $G$ and $H$ such that $J \subset G$ and $K \subset H$.

Remark 5.2: Every normal space is $C$-normal but not conversely.

Theorem 5.3: For a topological space $X$, the following properties are equivalent:

1. $X$ is $C$-normal;
2. for any disjoint $J$, $K \subset C$-$C(X)$, there exist disjoint $C^g$-open sets $G$, $H$ such that $J \subset G$ and $K \subset H$;
3. for any $J \subset C$-$C(X)$ and any $H \subset C$-$O(X)$ containing $J$, there exists a $C^g$-open set $G$ of $X$ such that $J \subset G \subset C^g$-$cl(G) \subset H$;
4. for any $J \subset C$-$C(X)$ and any $H \subset C$-$O(X)$ containing $J$, there exists an open set $G$ of $X$ such that $J \subset G \subset cl(G) \subset H$;
5. for any disjoint $J$, $K \subset C$-$C(X)$, there exist disjoint regular open sets $G$, $H$ such that $J \subset G$ and $K \subset H$.

Proof: (1) $\Rightarrow$ (2): Since every open set is $C^g$-open, the proof is obvious.
(2) \(\Rightarrow\) (3): Let \(J \in C-C(X)\) and \(H\) be any \(C\)-open set containing \(J\). Then \(J, X - H \in C-C(X)\) and \(J \cap (X - H) = \emptyset\). By (2), there exist \(Cg\)-open sets \(G, F\) such that \(J \subset G\), \(X - H \subset F\) and \(G \cap F = \emptyset\). Therefore, we have \(J \subset G \subset (X - F) \subset H\). Since \(G\) is \(Cg\)-open and \(X - F\) is \(Cg\)-closed, we obtain \(J \subset G \subset \text{Cg-cl}(G) \subset (X - F) \subset H\).

(3) \(\Rightarrow\) (4): Let \(J \in C-C(X)\) and \(J \subset H \in C-O(X)\). By (3), there exists a \(Cg\)-open set \(G_0\) of \(X\) such that \(J \subset G_0 \subset \text{Cg-cl}(G_0) \subset H\). Since \(\text{Cg-cl}(G_0)\) is \(Cg\)-closed and \(H \in C-O(X)\), \(\text{cl}(\text{Cg-cl}(G_0)) \subset H\). Put \(\text{int}(G_0) = G\), then \(G\) is open and \(J \subset G \subset \text{cl}(G) \subset H\).

(4) \(\Rightarrow\) (5): Let \(J, K\) be disjoint \(C\)-closed sets of \(X\). Then \(J \subset (X - K) \in C-O(X)\) and by (4) there exists an open set \(G_0\) such that \(J \subset G_0 \subset \text{cl}(G_0) \subset (X - K)\). Therefore, \(H_0 = (X - \text{cl}(G_0))\) is an open set such that \(J \subset G_0, K \subset H_0\) and \(G_0 \cap H_0 = \emptyset\). Moreover, put \(G = \text{int}(\text{cl}(G_0))\) and \(H = \text{int}(\text{cl}(H_0))\), then \(G, H\) are regular open sets such that \(J \subset G, K \subset H\) and \(G \cap H = \emptyset\).

(5) \(\Rightarrow\) (1): This is obvious.

We get a characterization of normal spaces by using \(Cg\)-open sets.

**Theorem 5.4:** For a topological space \(X\), the following properties are equivalent:

1. \(X\) is normal;
2. for any disjoint closed sets \(J\) and \(K\), there exist disjoint \(Cg\)-open sets \(G\) and \(H\) such that \(J \subset G\) and \(K \subset H\);
3. for any closed set \(J\) and any open set \(H\) containing \(J\), there exists a \(Cg\)-open set \(G\) of \(X\) such that \(J \subset G \subset \text{cl}(G) \subset H\).

**Proof:** (1) \(\Rightarrow\) (2): This is obvious since every open set is \(Cg\)-open.

(2) \(\Rightarrow\) (3): Let \(J\) be a closed set and \(H\) be any open set containing \(J\). Then \(J\) and \((X - H)\) are disjoint closed sets. There exist disjoint \(Cg\)-open sets \(G\) and \(F\) such that \(J \subset G\) and \((X - H) \subset F\). Since \(X - H\) is closed, we have \((X - H) \subset \text{int}(F)\) and \(G \cap \text{int}(F) = \emptyset\). Therefore, we obtain \(\text{cl}(G) \cap \text{int}(F) = \emptyset\) and hence \(J \subset G \subset \text{cl}(G) \subset (X - \text{int}(F)) \subset H\).

(3) \(\Rightarrow\) (1): Let \(J, K\) be disjoint closed sets of \(X\). Then \(J \subset (X - K)\) and \((X - K)\) is open. By (3), there exists a \(Cg\)-open set \(F\) of \(X\) such that \(J \subset F \subset \text{cl}(F) \subset (X - K)\). Since \(J\) is closed, we have \(J \subset \text{int}(F)\). Put \(G = \text{int}(F)\) and \(H = (X - \text{cl}(F))\). Then \(G\) and \(H\) are disjoint open sets of \(X\) such that \(J \subset G\) and \(K \subset H\). Hence, \(X\) is normal.

**Lemma 5.5:** A subset \(G\) of a space \(X\) is \(Cg\)-open if and only if \(F \subset \text{int}(G)\) whenever \(F \subset G\) and \(F\) is \(C\)-closed.

**Proof:** Let \(G\) be a \(Cg\)-open set then \(X - G\) is \(Cg\)-closed set. Since \(X - G\) is \(Cg\)-closed iff \(\text{cl}(X - G) \subset X - F\) whenever \(X - G \subset X - F\) and \(X - F\) is \(Cg\)-open, this implies that \(X - \text{int}(G) \subset X - F\) whenever \(F \subset G\) and \(F\) is \(Cg\)-closed (because \(\text{cl}(X - G) = X - \text{int}(G))\), i.e. \(F \subset \text{int}(G)\) whenever \(F \subset G\) and \(F\) is \(Cg\)-closed.

**Theorem 5.6:** For a space topological \(X\), the following are equivalent:

1. \(X\) is \(C\)-normal.
2. For any disjoint \(C\)-closed sets \(J\) and \(K\), there exist disjoint \(g\)-open sets \(G\) and \(H\) such that \(J \subset G\) and \(K \subset H\).
(3) For any disjoint C-closed sets J and K, there exist disjoint Cg-open sets G and H such that J ⊆ G and K ⊆ H.
(4) For any C-closed set J and any C-open set H containing J, there exists a g-open set G of X such that J ⊆ G ⊆ cl(G) ⊆ H.
(5) For any C-closed set J and any C-open set H containing J, there exists a Cg-open set G of X such that J ⊆ G ⊆ cl(G) ⊆ H.

Proof: (1) ⇒ (2): Let X be C-normal space. Let J, K be disjoint C-closed sets of X. By assumption, there exist disjoint open sets G, H such that J ⊆ G and K ⊆ H. Since every open set is g-open, so G and H are g-open sets such that J ⊆ G and K ⊆ H.

(2) ⇒ (3): Let J and K be two disjoint C-closed sets. By assumption, there exist disjoint g-open sets G and H such that J ⊆ G and K ⊆ H. Since every g-open set is Cg-open, G and H are Cg-open sets such that J ⊆ G and K ⊆ H.

(3) ⇒ (4): Let J be any C-closed set and H be any C-open set containing J. By assumption, there exist disjoint Cg-open sets G and H such that J ⊆ G and X − H ⊆ H. By Lemma 5.5, we get X − H ⊆ int(H) and cl(G) ∩ int(H) = φ. Hence J ⊆ G ⊆ cl(G) ⊆ X − int(H) ⊆ H.

(4) ⇒ (5): Let J be any C-closed set and H be any C-open set containing J. By assumption, there exist g-open set G of X such that J ⊆ G ⊆ cl(G) ⊆ H. Since every g-open set is Cg-open, there exists Cg-open sets G of X such that J ⊆ G ⊆ cl(G) ⊆ H.

(5) ⇒ (1): Let J, K be any two disjoint C-closed sets of X. Then J ⊆ X − K and X − K is C-open. By assumption, there exists Cg-open set G of X such that J ⊆ G ⊆ cl(G) ⊆ X − K. Put G = int(G), H = X - cl(G). Then G and H are disjoint open sets of X such that J ⊆ G and K ⊆ H.

Theorem 5.6: Let X be a C-normal space. Then a semi-regular subspace Y of X is also C-normal.
Proof: Let X be a C-normal space and Y be a semi-regular subspace of X. Let J ⊆ C-C(Y) and H ⊆ C-O(Y) containing J. Since Y is semi regular, so J ∈ C-C(X) and H ∈ C-O(X). Hence by Theorem 5.3(4), there exists an open set G in X such that J ⊆ G ⊆ clX(G) ⊆ H. This gives J ⊆ (G ∩ Y) ⊆ clY(G ∩ Y) ⊆ H, where G ∩ Y is open in Y and hence Y is C-normal.

6. FUNCTIONS AND C-NORMAL SPACES

Definition 6.1: A function f : X → Y is said to be:
(1) almost Cg-continuous if for any regular open set U of Y, f−1(U) ∈ Cg-O(X);
(2) almost Cg-closed if for any regular closed set J of X, f(J) ∈ Cg-C(Y).

Definition 6.2: A function f : X → Y is said to be:
(1) C-irresolute (resp. C-continuous [2]) if for any C-open (resp. open) set U of Y, f−1(U) is C-open in X;
(2) pre-C-closed (resp. C-closed [2]) if for any C-closed (resp. closed) set J of X, f(J) is C-closed in Y.
Theorem 6.3: A function \( f : X \to Y \) is an almost \( C_g \)-closed surjection iff for each subset \( P \) of \( Y \) and each regular open set \( G \) containing \( f^{-1}(P) \), there exists a \( C_g \)-open set \( H \) such that \( P \subseteq H \) and \( f^{-1}(H) \subseteq G \).

**Proof:** **Necessity.** Suppose that \( f \) is almost \( C_g \)-closed. Let \( P \) be a subset of \( Y \) and \( G \) be a regular open set of \( X \) containing \( f^{-1}(P) \). Put \( H = Y - f(X - G) \), then \( H \) is a \( C_g \)-open set of \( Y \) such that \( P \subseteq H \) and \( f^{-1}(H) \subseteq G \).

**Sufficiency:** Let \( J \) be any regular closed set of \( X \). Then \( f^{-1}(Y - f(J)) \subseteq (X - J) \) and \( X - J \) is regular open. There exists a \( C_g \)-open set \( H \) of \( Y \) such that \((Y - f(J)) \subseteq H \) and \( f^{-1}(H) \subseteq (X - J) \). Therefore, we have \( f(J) \supseteq (Y - H) \) and \( J \subseteq f^{-1}(Y - H) \). Hence, we obtain \( f(J) = Y - H \) and \( f(J) \) is \( C_g \)-closed in \( Y \). Therefore \( f \) is almost \( C_g \)-closed.

Theorem 6.4: If \( f : X \to Y \) is an almost \( C_g \)-closed \( C \)-irresolute (resp. \( C \)-continuous) surjection and \( X \) is \( C \)-normal, then \( Y \) is \( C \)-normal (resp. \( C \)-normal).

**Proof:** Let \( J \) and \( K \) be any disjoint \( C \)-closed (resp. closed) sets of \( Y \). Then \( f^{-1}(J) \) and \( f^{-1}(K) \) are disjoint \( C \)-closed sets of \( X \). Since \( X \) is \( C \)-normal, there exist disjoint open sets \( G \) and \( H \) of \( X \) such that \( f^{-1}(J) \subseteq G \) and \( f^{-1}(K) \subseteq H \). Put \( G_1 = \text{int}(\text{cl}(G)) \) and \( H_1 = \text{int}(\text{cl}(H)) \), then \( G_1 \) and \( H_1 \) are disjoint regular open sets of \( X \) such that \( f^{-1}(J) \subseteq G_1 \) and \( f^{-1}(K) \subseteq H_1 \). By Theorem 6.3, there exist \( C_g \)-open sets \( L \) and \( M \) of \( Y \) such that \( J \subseteq L \), \( K \subseteq M \). \( f^{-1}(L) \subseteq G_1 \) and \( f^{-1}(M) \subseteq H_1 \). Since \( G_1 \) and \( H_1 \) are disjoint, so \( L \) and \( M \) are also disjoint. It follows from Theorem 5.3 (resp. Theorem 5.4) that \( Y \) is \( C \)-normal (resp. \( C \)-normal).

Theorem 6.5: If \( f : X \to Y \) is a continuous almost \( C_g \)-closed surjection and \( X \) is a normal space, then \( Y \) is normal.

**Proof:** The proof is similar to that of Theorem 6.4.

Theorem 6.6: If \( f : X \to Y \) is an almost \( C_g \)-continuous pre-\( C \)-closed (resp. \( C \)-closed) injection and \( Y \) is \( C \)-normal, then \( X \) is \( C \)-normal (resp. \( C \)-normal).

**Proof:** Let \( J \) and \( K \) be disjoint \( C \)-closed (resp. closed) sets of \( X \). Since \( f \) is a pre-\( C \)-closed (resp. \( C \)-closed) injection, \( f(J) \) and \( f(K) \) are disjoint \( C \)-closed sets of \( Y \). Since \( Y \) is \( C \)-normal, there exist disjoint open sets \( G \) and \( H \) such that \( f(J) \subseteq G \) and \( f(K) \subseteq H \). Now, put \( G_1 = \text{int}(\text{cl}(G)) \) and \( H_1 = \text{int}(\text{cl}(H)) \), then \( G_1 \) and \( H_1 \) are disjoint regular open sets such that \( f(J) \subseteq G_1 \) and \( f(K) \subseteq H_1 \). Since \( f \) is almost \( C_g \)-continuous, \( f^{-1}(G_1) \) and \( f^{-1}(H_1) \) are disjoint \( C_g \)-open sets such that \( J \subseteq f^{-1}(G_1) \) and \( K \subseteq f^{-1}(H_1) \). It follows from Theorem 5.3 (resp. Theorem 5.4) that \( X \) is \( C \)-normal (resp. \( C \)-normal).

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