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POINTWISE V-SEMI-SLANT SUBMERSIONS

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Abstract

In this paper, we define pointwise v-semi-slant submersions fromalmost Hermitian manifolds onto Riemannian manifolds. The geometry of leaves of distributions which are associated with the definition of such maps is studied. The conditions for above submersions to be integrable and totally geodesic are also obtained in the paper. Finally, we provide an example ofsuch pointwise v-semi-slant submersion.

Key words and phrases: Kähler manifolds, Riemannian Submersions, Pointwise v-semi-slant submersions.

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Introductions

In differential geometry, the notion of Riemannian submersion was first studiedby O'Neill [14] and Gray [6]. Watson defined almost Hermitian submersions between Hermitian manifolds and he also showed that the base manifold and each fiber have the same kind of structure as the total space in most case [25]. Recently, according to the different conditions on Riemannian submersion, many authors have carried out several studies (like [8], [9], [10], [15], [17], [18], [19], [21], [22]). Lee and Sahin investigated pointwise slant submersions [11]. As a generalization of slant submersions, Sepet and Bozok defined pointwise semi-slant submersions from Hermitian manifolds onto Riemannian manifolds [23] and pointwise bi-slant submersions in [24]. Also, in [16], Park studied v-semi-slant submersions from Hermitian manifolds onto Riemannian manifolds and obtained some characterizations.On the other hand, it is well known that Riemannian submersions are related with physics and have their applications in the Yang Mills theory [4], Kaluza Klein theory [5], supergravity and superstring theories [7] etc. Some other applications of Riemannian submersions are statistica machine learning process, medical imaging [13], statistical analysis on manifolds [3] and robotic theory [1].

In this paper, we study pointwise v-semi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds. We investigate the integrability of distributions and the geometry of fibers. Also we obtain necessary and sufficient conditions for such maps to be totally geodesic and provide an example ofsuch submersion.

Preliminaries

Let M be an even-dimensional differentiable manifold. Let J be a (1,1) tensor field on M such that $J^2 = -I$, where I is identity operator. Then J is called an almost complex structure on M. The manifold M with an almost complex structure J is called an almost complex manifold [26]. It is well known that an almost complex manifold is necessarily orientable. Nijenhuis tensor N of an almost complex structure is defined as:

$$N(X_1, X_2) = [JX_1, JX_2] - [X_1, X_2] - J[JX_1, X_2] - J[X_1, JX_2],$$

for all $X_1, X_2 \in \Gamma(TM)$.

If Nijenhuis tensor field N on an almost complex manifold M is zero, then the almost complex manifold M is called a complex manifold.

Let g_M is a Riemannian metric on M such that

$$g_M(JX_1, JX_2) = g_M(X_1, X_2), \tag{2.1}$$

for all $X_1, X_2 \in \Gamma(TM)$.

Then g_M is called an almost Hermitian metric on M and manifold M with Hermitian metric g_M is called almost Hermitian manifold. The Riemannian connection ∇ of the almost Hermitian manifold M can be extended to the whole tensor algebra on M. Tensor fields $(\nabla_{Y_1}J)Y_2$ is defined as

$$\left(\nabla_{Y_1}J\right)Y_2 = \nabla_{Y_1}JY_2 - J\nabla_{Y_1}Y_2, \tag{2.2}$$

for all $Y_1, Y_2 \in \Gamma(TM)$.
An almost Hermitian manifold (M, g_M, J) is called a Kähler manifold if
Then (M, g_M, J) is said to be an almost Hermitian manifold, and if

$$(\nabla_{X_1}J)\, X_2 = 0, \tag{2.3}$$

for all $X_1, X_2 \in \Gamma(TM)$, then $(M, g_M, J)$is said to be a Kähler manifold, where ∇ is the Levi-Civita connection on M.
Let F: $(M, g_M) \longrightarrow (N, g_N)$ be a Riemannian submersion ([12], [20]). Define O'Neill's tensors T and A [14] by

$$A_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2, \tag{2.4}$$

$$T_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2, \tag{2.5}$$

for any $E_1, E_2 \in \Gamma(TM)$.
It is easy to see that T_{E_1} and A_{E_1} are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. From equations (2.4) and (2.5), we have

$$\nabla_{X_1}X_2 = T_{X_1}X_2 + \mathcal{V}\nabla_{X_1}X_2, \tag{2.6}$$

$$\nabla_{X_1}Z_1 = T_{X_1}Z_1 + \mathcal{H}\nabla_{X_1}Z_1, \tag{2.7}$$

$$\nabla_{Z_1}X_1 = A_{Z_1}X_1 + \mathcal{V}\nabla_{Z_1}X_1, \tag{2.8}$$

$$\nabla_{Z_1}Z_2 = A_{Z_1}Z_2 + \mathcal{H}\nabla_{Z_1}Z_2, \tag{2.9}$$

for all $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$, where $\mathcal{H}\nabla_{X_1}Z_1 = A_{Z_1}X_1$, if Z_1 is basic. Let (M, g_M) and (N, g_N) be Riemannian manifolds and F: $(M, g_M) \rightarrow (N, g_N)$ be a C^{∞}-map then the second fundamental form of F is given by

$$(\nabla F_*)\,(X_1, X_2) = \nabla^F_{X_1} F_*\,(X_2) - F_*(\nabla^M_{X_1}X_2) \tag{2.10}$$

for $X_1, X_2 \in \Gamma(TM)$, where ∇^F is the pullback connection, and ∇ is the Riemannian connections of the metric g_M.

In addition, a differentiable map F between two Riemannian manifolds is totally geodesic [2] if

$$(\nabla F_*)(X_1, X_2) = 0\,, \tag{2.11}$$

for $X_1, X_2 \in \Gamma(TM)$.

Lemma 1. [2] Let $((M, g_M)$ and (N, g_N) are Riemannian manifolds. If F: $(M, g_M) \longrightarrow (N, g_N)$ be a Riemannian submersion, then for any horizontal vector fields Y_1, Y_2 and vertical vector fields W_1, W_2, we have

(i) $(\nabla F_*)(Y_1, Y_2) = 0$,
(ii) $(\nabla F_*)(W_1, W_2) = -F_*(T_{W_1} W_2) = -F_*(\nabla_{W_1} W_2)$,
(iii) $(\nabla F_*)(Y_1, W_1) = -F_*(A_{Y_1} W_1) = -F_*(\nabla_{Y_1} W_1)$.

Pointwise V-semi-slant submersions

In this section, pointwise v-semi-slant submersions from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) is defined and studied.

We now present the notion of pointwise v-semi-slant submersions as follows:

Definition 1.A Riemannian submersion F: $(M, g_M, J) \longrightarrow (N, g_N)$ is called apointwise v-semi-slant submersion if there is a distribution $\Gamma(\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = D_1 \oplus D_2, \qquad J(D_1) = D_{1,}$$

and for $p \in M$ and $Z \in (D_2)_P$, the angle $\theta = \theta(Z)$ between JZ and the space $(D_2)_P$ is independent of the choice of the nonzero vector Z, where D_2 is the orthogonal complement of D_1 in $(\ker F_*)^\perp$. The angle θ is called pointwise v-semi-slant function of the slant submersion.

Let F be a pointwise v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, we have

$$TM = (\ker F_*) \oplus (\ker F_*)^\perp. \tag{3.1}$$

Further, we put

$$Z_1 = PZ_1 + QZ_1 \tag{3.2}$$

for any vector field $Z_1 \in \Gamma(\ker F_*)^\perp$, where P and Q are projection morphisms of $\Gamma(\ker F_*)^\perp$ onto D_1 and D_2, respectively.

For U $\in \Gamma(\ker F_*)^\perp$, we get

$$JU = BU + CU \tag{3.3}$$

where $BU \in \Gamma(\ker F_*)$ and $CU \in \Gamma(\ker F_*)^\perp$. Also, for any $W \in \Gamma(\ker F_*)$, we have

$$JW = \phi W + \omega W \tag{3.4}$$

Where $\phi W \in \Gamma(\ker F_*)$ and $\omega W \in \Gamma(\ker F_*)^{\perp}$.

Lemma 2. Let F be a pointwise v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, we have

$$\phi^2 Z_1 + B\omega Z_1 = -Z_1, \omega\phi Z_1 + C\omega Z_1 = 0,$$

$$\omega B Z_2 + C^2 Z_2 = -Z_2, \phi B Z_2 + B C Z_2 = 0$$

for any $Z_1 \in \Gamma(\ker F_*)$ and $Z_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. With the help of equations (3.3), (3.4) along with the condition $J^2 = -I$ we obtain the Lemma 2.

Lemma 3. Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) Riemannian manifold. F: $(M, g_M, J) \longrightarrow (N, g_N)$ is a pointwise v-semi-slant submersion ifand only if

$$C^2 V = -(cos^2\theta)V,$$

for $V \in \Gamma(D_2)$.

Proof.The proof of Lemma 3 is the same as that one for v-semi-slant submersion see proposition (3.5) and remark (3.6) of [16]. So we omit it.

Lemma 4. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, we have

$$\mathcal{V}\nabla_{U_1}\phi V_2 + \mathcal{T}_{U_1}\omega V_2 = \phi\mathcal{V}\nabla_{U_1}V_2 + B\mathcal{T}_{U_1}V_2, \quad (3.5)$$

$$\mathcal{T}_{U_1}\phi V_2 + \mathcal{H}\nabla_{U_1}\omega V_2 = \omega\mathcal{V}\nabla_{U_1}V_2 + C\mathcal{T}_{U_1}V_2, \quad (3.6)$$

$$\mathcal{V}\nabla_{X_1}BY_2 + \mathcal{A}_{X_1}CY_2 = \phi\mathcal{A}_{X_1}Y_2 + B\mathcal{H}\nabla_{X_1}Y_2, \quad (3.7)$$

$$\mathcal{A}_{X_1}BY_2 + \mathcal{H}\nabla_{X_1}CY_2 = \omega\mathcal{A}_{X_1}Y_2 + C\mathcal{H}\nabla_{X_1}Y_2, \quad (3.8)$$

$$\mathcal{V}\nabla_{U_1}BX_1 + \mathcal{T}_{U_1}CX_1 = \phi\mathcal{T}_{U_1}X_1 + B\mathcal{H}\nabla_{U_1}X_1, \quad (3.9)$$

$$\mathcal{T}_{U_1}BX_1 + \mathcal{H}\nabla_{U_1}CX_1 = \omega\mathcal{T}_{U_1}X_1 + C\mathcal{H}\nabla_{U_1}X_1, \quad (3.10)$$

$$\mathcal{V}\nabla_{X_1}\phi U_1 + \mathcal{A}_{X_1}\omega U_1 = B\mathcal{A}_{X_1}U_1 + \phi\mathcal{V}\nabla_{X_1}U_1, \quad (3.11)$$

$$\mathcal{A}_{X_1}\phi U_1 + \mathcal{H}\nabla_{X_1}\omega U_1 = C\mathcal{A}_{X_1}U_1 + \omega\mathcal{V}\nabla_{X_1}U_1, \quad (3.12)$$

for any $U_1, V_2 \in \Gamma(\ker F_*)$ and $X_1, Y_2 \in \Gamma(\ker F_*)^{\perp}$).

Proof. By equations (2.6)-(2.9),(3.3) and (3.4), we get equations (3.5)-(3.12).

Now, we define

$$(\nabla_{U_1}\phi)U_2 = \mathcal{V}\nabla_{U_1}\phi U_2 - \phi\mathcal{V}\nabla_{U_1}U_2, \quad (3.13)$$

$$(\nabla_{U_1}\omega)U_2 = \mathcal{H}\nabla_{U_1}\omega U_2 - \omega\mathcal{V}\nabla_{U_1}U_2, \quad (3.14)$$

$$(\nabla_{V_1}C)V_2 = \mathcal{H}\nabla_{V_1}CV_2 - C\mathcal{H}\nabla_{V_1}V_2, \quad (3.15)$$

$$(\nabla_{V_1}B)V_2 = \mathcal{V}\nabla_{V_1}BV_2 - B\mathcal{H}\nabla_{V_1}V_2 \quad (3.16)$$

for any $U_1, U_2 \in \Gamma(\ker F_*)$ and $V_1, V_2 \in \Gamma(\ker F_*)^{\perp}$).

Lemma 5. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, we have

$$(\nabla_{U_1}\phi)U_2 = BT_{U_1}U_2 - T_{U_1}\omega U_2,$$

$$(\nabla_{U_1}\omega)U_2 = C\mathcal{T}_{U_1}U_2 - \mathcal{T}_{U_1}\phi U_2,$$

$$(\nabla_{V_1}C)V_2 = \boldsymbol{\omega}A_{V_1}V_2 - A_{V_1}BV_2,$$

$$(\nabla_{V_1}B)V_2 = \boldsymbol{\phi}A_{V_1}V_2 - A_{V_1}CV_2$$

for any $U_1, U_2 \in \Gamma(\ker F_*)$ and $V_1, V_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. On the account of equations (3.5)-(3.8) and (3.13)-(3.16), we obtain required result of Lemma 5.

Consequently, if $\boldsymbol{\phi}$ and $\boldsymbol{\omega}$ are parallel tensor w.r.t. Levi-Civita connection ∇ defined on M, we get

$$BT_{U_1}U_2 = T_{U_1}\omega U_2, \qquad C\mathcal{T}_{U_1}U_2 = \mathcal{T}_{U_1}\phi U_2,$$

for any $U_1, U_2 \in \Gamma(TM)$.

Theorem 1. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, D_1 is integrable if and only if

$$\omega(A_{X_1}JX_2 - A_{X_2}JX_1) = C(\mathcal{H}\nabla_{X_2}JX_1 - \mathcal{H}\nabla_{X_1}JX_2),$$

for $X_1, X_2 \in \Gamma(D_1)$.

Proof. For $X_1, X_2 \in \Gamma(D_1)$ and $Z_1 \in \Gamma(D_2)$, using equations (2.1), (2.3), (2.9), (3:3) and (3.4), we have

$$g_M([X_1, X_2], Z_1) = g_M(\nabla_{X_1}JX_2, JZ_1) - g_M(\nabla_{X_2}JX_1, JZ_1),$$

$$g_M([X_1, X_2], Z_1) = g_M(\omega(A_{X_1}JX_2 - A_{X_2}JX_1), Z_1) -$$

$$g_M\big(C('\mathrm{H}\nabla_{X_2}JX_1 - '\mathrm{H}\nabla_{X_1}JX_2), Z_1\big),$$

which completes the proof.

Theorem 2. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, D_2 is integrable if and only if

$$g_M\big(A_{Z_1}BZ_2 - A_{Z_2}BZ_1, JV_1\big) = g_M\big(A_{Z_1}BCZ_2 - A_{Z_2}BCZ_1, V_1\big),$$

for $Z_1, Z_2 \in \Gamma(D_2)$ and $V_1 \in \Gamma(D_1)$.

Proof. For $Z_1, Z_2 \in \Gamma(D_2)$ and $V_1 \in \Gamma(D_1)$, we have

$$g_M([Z_1, Z_2], V_1) = g_M\big(\nabla_{Z_1}JZ_2, JV_1\big) - g_M\big(\nabla_{Z_2}JX_1, JV_1\big),$$

$$g_M([Z_1, Z_2], V_1) = cos^2\theta g_M([Z_1, Z_2], V_1) + g_M\big(A_{Z_1}BZ_2 - A_{Z_2}BZ_1, JV_1\big) - g_M\big(A_{Z_1}BCZ_2 - A_{Z_2}BCZ_1, V_1\big).$$

Now, we have

$$sin^2\theta g_M([Z_1, Z_2], V_1) = g_M\big(A_{Z_1}BZ_2 - A_{Z_2}BZ_1, JV_1\big) - g_M\big(A_{Z_1}BCZ_2 - A_{Z_2}BCZ_1, V_1\big),$$

from above the proof is completed.

Theorem 3. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). The distribution $(\ker F_*)^{\perp}$ becomesa totally geodesic foliation on M if and only if

$$sin^2\theta g_M([X_1, U_1], X_2) - cos^2\theta g_M\big('\mathrm{H}\nabla_{U_1}PX_1, X_2\big)$$

$$= -g_M\big('\mathrm{H}\nabla_{U_1}JPX_1, X_2\big) - g_M\big('\mathrm{H}\nabla_{U_1}JPX_1, X_2\big) - g_M\big(\mathrm{V}\nabla_{U_1}BQX_1, BX_2\big)$$

$$-g_M(T_{U_1}BQX_1, CX_2) + g_M(T_{U_1}BCQX_1, X_2) + sin2\theta U_1[\theta] g_M(QX_1, QX_2),$$

for $U_1 \in \Gamma(\ker F_*)$ and $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. For $U_1 \in \Gamma(\ker F_*)$ and $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$, using equations (2.1), (2.3),

(2.6), (2.7), (3.2), (3.3), (3.4) and Lemma 3, we have

$$g_M(\nabla_{X_1}X_2,U_1) = -g_M([X_1,U_1],\,X_2) - g_M(\nabla_{U_1}X_1,\,X_2),$$

$$= -g_M([X_1,U_1],\,X_2) - g_M(\nabla_{U_1}JPX_1,\,JX_2)$$

$$-g_M(\nabla_{U_1}BQX_1,J\,X_2) + g_M(\nabla_{U_1}BCQX_1,\,X_2)$$

$$-cos^2\theta g_M g_M(\nabla_{U_1}QX_1,\,X_2) + \,sin2\theta U_1[\theta]g_M(QX_1,\,QX_2).$$

Now, we obtain

$$sin^2\theta g_M(\nabla_{X_1}X_2,U_1)$$

$$= -sin^2\theta g_M([X_1,U_1],X_2) + cos^2\theta g_M(\mathcal{H}\nabla_{U_1}PX_1,X_2)$$

$$-g_M(\mathcal{H}\nabla_{U_1}JPX_1,X_2) - g_M(T_{U_1}JPX_1,X_2) - g_M(\mathcal{V}\nabla_{U_1}BQX_1,BX_2)$$

$$-g_M\,(T_{U_1}BQX_1,\,CX_2) + g_M(T_{U_1}BCQX_1,\,X_2)$$

$$+sin2\theta U_1[\theta]g_M(QX_1,\,QX_2).$$

Theorem 4.Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). The distribution $(\ker F_*)$ becomesa totally geodesic foliation on M if and only if

$$g_M(\mathcal{V}\nabla_{X_1}X_2,BCZ_1) = g_M(\mathcal{V}\nabla_{X_1}\phi X_2,BZ_1) + g_M(T_{X_1}\omega X_2,BZ_1),$$

for $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1 \in \Gamma(\ker F_*)^{\perp}$.

Proof. For $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1 \in \Gamma(\ker F_*)^{\perp}$, using equations (2.1), (2.3), (2.6), (2.7), (3.3) and Lemma 3, we have

$$g_M(\nabla_{X_1}X_2,Z_1) = g_M(\nabla_{X_1}JX_2,JZ_1),$$

$$g_M(\nabla_{X_1}X_2,Z_1) = g_M(\mathcal{V}\nabla_{X_1}\phi X_2,BZ_1) + g_M(T_{X_1}\omega X_2,BZ_1)$$

$$+cos^2\theta g_M(\nabla_{X_1}X_2,Z_1) - g_M(\mathcal{V}\nabla_{X_1}X_2,BCZ_1).$$

Now, we get

$$sin^2\theta g_M(\nabla_{X_1}X_2,Z_1) = g_M(\mathcal{V}\nabla_{X_1}\phi X_2,BZ_1) + g_M(T_{X_1}\omega X_2,BZ_1)$$

$$-g_M(\mathcal{V}\nabla_{X_1}X_2,BCZ_1).$$

Theorem 5. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). The distribution D_1becomesa totally geodesic foliation on M if and only if

$$g_M\left(A_{V_1}JV_2, BZ_1\right) = g_M\left(A_{V_1}V_2, BCZ_1\right),$$

$$g_M\left(A_{V_1}JV_2, \phi X_1\right) = -g_M\left(\mathcal{H}\nabla_{V_1}JV_2, \omega X_1\right),$$

for $V_1, V_2 \in \Gamma(D_1)$, $Z_1 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker F_*)$.

Proof. For $V_1, V_2 \in \Gamma(D_1)$, $Z_1 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker F_*)$, using equations (2.1), (2.3), (2.9), (3.3) and Lemma 3, we have

$$g_M\left(\nabla_{V_1}V_2, Z_1\right) = g_M\left(\nabla_{V_1}JV_2, BZ_1\right) - g_M\left(\nabla_{V_1}V_2, C^2Z_1\right) - g_M\left(\nabla_{V_1}V_2, BCZ_1\right),$$

$$= g_M\left(A_{V_1}JV_2, BZ_1\right) + cos^2\theta g_M\left(\nabla_{V_1}V_2, C^2Z_1\right) - g_M\left(A_{V_1}V_2, BCZ_1\right).$$

Now, we get

$$sin^2\theta g_M\left(\nabla_{V_1}V_2, Z_1\right) = g_M\left(A_{V_1}JV_2, BZ_1\right) - g_M\left(A_{V_1}V_2, BCZ_1\right).$$

Now, again using equations (2.1), (2.3), (2.9) and (3.4), we have

$$g_M\left(\nabla_{V_1}V_2, X_1\right) = g_M\left(\nabla_{V_1}JV_2, JX_1\right),$$

$$= g_M\left(\nabla_{V_1}JV_2, \phi X_1\right) + g_M\left(\nabla_{V_1}JV_2, \omega X_1\right)$$

$$= g_M\left(A_{V_1}JV_2, \phi X_1\right) + g_M\left(\mathcal{H}\nabla_{V_1}JV_2, \omega X_1\right).$$

this completes the proof.

Theorem 6. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). The distribution D_2 becomesa totally geodesic foliation on M if and only if

$$g_M\left(A_{W_1}BW_2, JX_1\right) = g_M\left(A_{W_1}BCW_2, X_1\right),$$

$$sin^2\theta g_M([W_1, X_2], W_2) = -g_M\left(T_{X_2}BW_1, CW_2\right) - g_M\left(\mathcal{V}\nabla_{X_2}BW_1, BW_2\right) +$$

$$sin2\theta X_2[\theta] g_M(W_1, W_2) + g_M\left(T_{X_2}BCW_1, W_2\right),$$

for $W_1, W_2 \in \Gamma(D_2)$, $X_1 \in \Gamma(D_1)$ and $X_2 \in \Gamma(\ker F_*)$.

Proof. For $W_1, W_2 \in \Gamma(D_2)$, $X_1 \in \Gamma(D_1)$ and $X_2 \in \Gamma(\ker F_*)$, using equations (2.1), (2.3), (2.8), (3.3) and Lemma 3, we have

$$g_M\left(\nabla_{W_1}W_2, X_1\right) = g_M\left(\nabla_{W_1}JW_2, JX_1\right),$$

$$= g_M\left(\nabla_{W_1}BW_2, JX_1\right) + cos^2\theta g_M\left(\nabla_{W_1}W_2, X_1\right)$$

$$-g_M\left(\nabla_{W_1} BCW_2, X_1\right).$$

Now, we have

$$sin^2\theta g_M\left(\nabla_{W_1} W_2, X_1\right) = g_M\left(A_{W_1} BW_2, JX_1\right) - g_M\left(A_{W_1} BCW_2, X_1\right).$$

Next, from equations (2.1), (2.3), (2.6), (3.3) and Lemma 3, we have

$$g_M\left(\nabla_{W_1} W_2, X_2\right) = -g_M([W_1, X_2], W_2) - g_M\left(\nabla_{X_2} W_1, W_2\right)$$

$$= -g_M([W_1, X_2], W_2) - g_M\left(\nabla_{X_2} BW_1, JW_2\right)$$

$$-cos^2\theta g_M\left(\nabla_{X_2} W_1, W_2\right) + sin2\theta X_2[\theta] g_M(W_1, W_2)$$

$$+g_M\left(\nabla_{X_2} BCW_1, W_2\right).$$

Now, we have

$$sin^2\theta g_M\left(\nabla_{W_1} W_2, X_2\right) = -sin^2\theta g_M([W_1, X_2], W_2) - g_M\left(T_{X_2} BW_1, CW_2\right)$$

$$-g_M\left(\mathcal{V}\nabla_{X_2} BW_1, BW_2\right) + sin2\theta X_2[\theta] g_M(W_1, W_2)$$

$$+g_M\left(T_{X_2} BCW_1, W_2\right).$$

Theorem 7. Let F be a pointwise v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then, F is a totally geodesic mapif and only if

$$CT_{Y_1}\phi Y_2 + \omega\mathcal{V}\nabla_{Y_1}\phi Y_2 + C'\mathcal{H}\nabla_{Y_1}\omega Y_2 + \omega T_{Y_1}\omega Y_2 = 0,$$

$$C'\mathcal{H}\nabla_{Y_1} JW_1 + \omega T_{Y_1} JW_1 = 0,$$

$$CT_{Y_1} BV_1 + \omega\mathcal{V}\nabla_{Y_1} BV_1 + T_{Y_1} BCV_1 - cos^2\theta'\mathcal{H}\nabla_{Y_1} V_1 + sin2\theta Y_1[\theta] V_1 = 0,$$

for $W_1 \in \Gamma(D_1)$, $V_1 \in \Gamma(D_2)$ and $Y_1, Y_2 \in \Gamma(\ker F_*)$.

Proof. Since F is a Riemannian map, we have

$$(\nabla F_*)(Z_1, Z_2) = 0,$$

for$Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp})$.

For $Y_1, Y_2 \in \Gamma(\ker F_*)$, using equations (2.3), (2.6), (2.7), (2.10), (3.3)and (3.4), we have

$$(\nabla F_*)(Y_1, Y_2) = -F_*\left(\nabla_{Y_1} Y_2\right),$$

$$= -F_*\left(JT_{Y_1}\phi Y_2 + J\mathcal{V}\nabla_{Y_1}\phi Y_2 + J'\mathcal{H}\nabla_{Y_1}\omega Y_2 + JT_{Y_1}\omega Y_2\right),$$

$$= -F_*(BT_{Y_1}\phi Y_2 + CT_{Y_1}\phi Y_2 + \phi\mathcal{V}\nabla_{Y_1}\phi Y_2 + \omega\mathcal{V}\nabla_{Y_1}\phi Y_2$$

$$+B'\mathrm{H}\nabla_{Y_1}\omega Y_2 + C'\mathrm{H}\nabla_{Y_1}\omega Y_2 + \phi T_{Y_1}\omega Y_2 + T_{Y_1}\omega Y_2).$$

For $Y_1 \in \Gamma(\ker F_*)$ and $W_1 \in \Gamma(D_1)$, using equations (2.3), (2.7), (2.10), (3.3) and (3.4), we have

$$(\nabla F_*)(Y_1, W_1) = -F_*\left(\nabla_{Y_1} W_1\right),$$

$$= F_*(B'\mathrm{H}\nabla_{Y_1}JW_1 + C'\mathrm{H}\nabla_{Y_1}JW_1 + \phi T_{Y_1}JW_1 + \omega T_{Y_1}JW_1).$$

For $Y_1 \in \Gamma(\ker F_*)$ and $V_1 \in \Gamma(D_2)$, using equations (2.3), (2.6), (2.7), (2.10), (3.3) and Lemma 3, we have

$$(\nabla F_*)(Y_1, V_1) = -F_*\left(\nabla_{Y_1} V_1\right),$$

$$= F_*(BT_{Y_1}BV_1 + CT_{Y_1}BV_1 + \phi \mathrm{V}\nabla_{Y_1}BV_1 + \omega \mathrm{V}\nabla_{Y_1}BV_1$$

$$+T_{Y_1}BCV_1 + \mathrm{V}\nabla_{Y_1}BCV_1 - cos^2\theta'\mathrm{H}\nabla_{Y_1}V_1$$

$$-cos^2\theta T_{Y_1}V_1 + sin2\theta Y_1[\theta]V_1).$$

Example

Let R^{2s} be Euclidean space. Let $(Y_1, Y_2, \dots\dots\dots.. Y_{2s-1}, Y_{2s})$ be the coordinates of R^{2s}. Define an almost complex structure J on R^{2s} as follows:

$$J(a_1\frac{\partial}{\partial Y_1} + a_2\frac{\partial}{\partial Y_2} + \dots\dots\dots\dots + a_{2s-1}\frac{\partial}{\partial Y_{2s-1}} + a_{2s}\frac{\partial}{\partial Y_{2s}})$$

$$= -a_2\frac{\partial}{\partial Y_1} + a_1\frac{\partial}{\partial Y_2} - \cdots\dots\dots - a_{2s}\frac{\partial}{\partial Y_{2s-1}} + a_{2s-1}\frac{\partial}{\partial Y_{2s}}$$

where $a_1, a_2, \dots\dots.. a_{2s-1}, a_{2s}$ are C^{∞}-functions on R^{2s}.

Example 1. Define a map F: $R^6 \longrightarrow R^2$

$$F(y_1, y_2, \dots.., y_6) = (y_1 sin \propto + y_3 cos \propto, y_4)$$

which is a pointwise v-semi-slant submersion such that

$$\Gamma(\ker F_*) =< cos \propto \frac{\partial}{\partial Y_1} - sin \propto \frac{\partial}{\partial Y_3}, \frac{\partial}{\partial Y_2}, \frac{\partial}{\partial Y_5}, \frac{\partial}{\partial Y_6} >,$$

$$(\ker F_*)^{\perp}) =< sin \propto \frac{\partial}{\partial Y_1} + cos \propto \frac{\partial}{\partial Y_3}, \frac{\partial}{\partial Y_4} >,$$

$$(\ker F_*)^{\perp}) = D_1 \oplus D_2,$$

where

$$D_1 =< \frac{\partial}{\partial Y_5}, \frac{\partial}{\partial Y_6} >, D_2 =< \cos\propto \frac{\partial}{\partial Y_1} - \sin\propto \frac{\partial}{\partial Y_3}, \frac{\partial}{\partial Y_2} >.$$

Thus is a pointwise v-semi-slant submersion with slant functions θ=∝.

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