\( \pi gN^* \)-CLOSED SETS AND QUASI \( N^* \)-NORMAL SPACES

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Abstract. In this paper, we introduce a new class of sets called \( gN^* \)-closed, \( \pi gN^* \)-closed sets and its properties are studied and we introduce a new concept of quasi-normal spaces called quasi \( N^* \)-normal spaces by using \( N^* \)-open sets due to G. Navalagi [6] in topological spaces and obtained several properties of such a space. Further we obtain a characterization and preservation theorems for quasi \( N^* \)-normal spaces and by using \( N^* \)-open sets.

1. Introduction

The notion of quasi normal space was introduced by Zaitsev [11], Dontchev and Noiri [2] introduce the notion of \( \pi g \)-closed sets as a weak form of \( g \)-closed sets due to Levine [4]. By using \( \pi g \)-closed sets, Dontchev and Noiri [2] obtained a new characterization of quasi normal spaces. G. Navalagi [6] introduced the concept of \( N^* \) and \( *N \)-closed sets and discuss some of their basic properties. Recently, Jeyanthi and Janaki [3] introduced the concepts of quasi r-normal spaces in topological spaces by using regular open sets in topological spaces and obtained some characterizations and preservation theorems of such spaces. We introduce the notion of \( N^*g \)-closed, \( N^*\alpha g \)-open, \( \pi gN^* \)-closed, \( \pi gN^* \)-open sets, \( \pi gN^* \)-closed, almost \( \pi gN^* \)-closed, \( \pi gN^* \)-continuous and almost \( \pi gN^* \)-continuous functions and its properties are studied. Further we obtain characterization and preservation theorems for quasi \( N^* \)-normal spaces.

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2. Preliminaries.

2.1. Definition. A subset \( A \) of a topological space \( X \) is called

1. regular closed [11] if \( A = \text{cl}(\text{int}(A)) \).

2. generalized closed [4] (briefly, g-closed) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \).

3. \( \pi g \)-closed [2] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \pi \)-open in \( X \).

4. \( \alpha \)-closed [8] if \( \text{cl}(\text{int}(\text{cl}(A))) \subseteq A \).

5. \( \alpha g \)-closed [5] if \( \alpha - \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is in \( X \).

6. \( \pi g \alpha \)-closed [1] if \( \alpha - \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \pi \)-open in \( X \).

The finite union of regular open sets is said to be \( \pi \)-open. The complement of \( \pi \)-open set is said to be \( \pi \)-closed. The complement of regular closed (resp. g-closed, \( \pi g \)-closed, \( \alpha \)-closed, \( \alpha g \)-closed, \( \pi g \alpha \)-closed) set is said to be regular open (resp. g-open, \( \pi g \)-open, \( \alpha \)-open, \( \alpha g \)-open, \( \pi g \alpha \)-open) sets.

2.2. Definition. A subset \( A \) of a topological space \( X \) is called

1. \( \mathbb{N}^* \)-closed [6] if \( \alpha g - \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \), and \( U \) is \( \mathbb{N}^* \)-open in \( X \).

2. \( \mathbb{N}^g* \)-closed if \( \mathbb{N}^* - \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \), and \( U \) is open in \( X \).

3. \( \pi g \mathbb{N}^* \)-closed if \( \mathbb{N}^* - \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \pi \)-open in \( X \).

The complement of \( \mathbb{N}^* - \) closed (resp. \( \mathbb{N}^g* - \) closed, \( \pi g \mathbb{N}^* - \) closed) sets is said to be \( \mathbb{N}^* \)-open (resp. \( \mathbb{N}^g* \)-open, \( \pi g \mathbb{N}^* \)-open). The intersection of all \( \mathbb{N}^* - \) closed subsets of \( X \) containing \( A \) (i.e. super sets of \( A \)) is called the \( \mathbb{N}^* \)-closure of \( A \) and is denoted by \( \mathbb{N}^* - \text{cl} (A) \). The union of all \( \mathbb{N}^* - \) open sets contained in \( A \) is called \( \mathbb{N}^* \)-interior of \( A \) and is denoted by \( \mathbb{N}^* - \text{int}(A) \). The family of all \( \mathbb{N}^* - \) open (resp. \( \mathbb{N}^* - \) closed) sets of a space \( X \) is denoted by \( \mathbb{N}^* \mathcal{O}(X) \) (resp. \( \mathbb{N}^* \mathcal{C}(X) \)).

2.3. Lemma. Let \( X \) be a topological space. Then

1. Every \( \alpha \)-closed subset of \( X \) is \( \mathbb{N}^* \)-closed

2. Every \( \alpha \)-open subset of \( X \) is \( \mathbb{N}^* \)-open.
We have the following implications for the properties of subsets.

\[
\begin{align*}
\text{closed} & \Rightarrow \text{g-closed} & \Rightarrow \pi\text{g-closed} \\
\downarrow & & \downarrow \\
\alpha\text{-closed} & \Rightarrow \alpha\text{g-closed} & \Rightarrow \pi\alpha\text{g-closed} \\
\downarrow & & \downarrow \\
N^*\text{-closed} & \Rightarrow N^*\text{g-closed} & \Rightarrow \pi\text{gN}^*\text{-closed}
\end{align*}
\]

Where none of the implications is reversible as can be seen from the following examples.

\textbf{2.4. Example.} Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{d\}, \{a, d\}, \{a, b, c, d\}, \{a, b, d\}, \{a, d\}, X\} \). Then the set \( A = \{a\} \) is \( \pi\alpha\text{g-closed} \) set as well as \( \pi\text{gN}^*\text{-closed} \) set but not g-closed set in \( X \).

\textbf{2.5. Example.} Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, d, c\}, \{a, b, d\}, \{a, b, c\}, X\} \). Then the set \( A = \{c\} \) is \( \pi\alpha\text{g-closed} \) as well as \( \pi\text{gN}^*\text{-closed} \) but not \( \alpha\text{g-closed} \) and not \( N^*\text{g-closed} \) set in \( X \).

\textbf{2.6. Example.} Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{d\}, \{a, d\}, \{a, b, c, d\}, \{a, b, d\}, \{a, d\}, X\} \). Then the set \( A = \{c\} \) is \( \pi\alpha\text{g-closed} \) as well as \( \pi\text{gN}^*\text{-closed} \) but not \( \pi\text{g-closed} \) set in \( X \).

\textbf{2.7. Theorem.}

(a) Finite union of \( \pi\text{gN}^*\text{-closed} \) sets are \( \pi\text{gN}^*\text{-closed} \).

(b) Finite intersection of \( \pi\text{gN}^*\text{-closed} \) need not be a \( \pi\text{gN}^*\text{-closed} \).

(c) A countable union of \( \pi\text{gN}^*\text{-closed} \) sets need not be a \( \pi\text{gN}^*\text{-closed} \).

\textbf{Proof.} (a) Let \( A \) and \( B \) be \( \pi\text{gN}^*\text{-closed} \) sets. Therefore \( N^*\text{-cl} (A) \subseteq U \) and \( N^*\text{-cl}(B) \subseteq U \) whenever \( A \subseteq U, B \subseteq U \) and \( U \) is \( \pi\text{-open} \). Let \( A \cup B \subseteq U \) where \( U \) is \( \pi\text{-open} \). Since \( N^*\text{-cl}(A \cup B) \subseteq N^*\text{-cl}(A) \cup N^*\text{-cl}(B) \subseteq U \), we have \( A \cup B \) is \( \pi\text{gN}^*\text{-closed} \).

(b) Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, \{a, d\}, \{a, b, d\}\} \). Let \( A = \{a, b, c\}, B = \{a, b, d\} \). \( A \) and \( B \) are \( \pi\text{gN}^*\text{-closed} \) sets. But \( A \cap B = \{a, b\} \subseteq \{a, b\} \) which is \( \pi\text{-open} \). \( N^*\text{-cl}(A \cap B) \not\subseteq \{a, b\} \). Hence \( A \cap B \) is not \( \pi\text{gN}^*\text{-closed} \).

(c) Let \( R \) be the real line with the usual topology. Every singleton is \( \pi\text{gN}^*\text{-closed} \). But, \( A = \{1/i : i = 2, 3, 4 \ldots\} \) is not \( \pi\text{gN}^*\text{-closed} \). Since \( A \subseteq (0, 1) \) which is \( \pi\text{-open} \) but \( N^*\text{-cl}(A) \not\subseteq (0, 1) \).
2.8. **Theorem.** If $A$ is $\pi g N^*$-closed and $A \subseteq B \subseteq N^* - \text{cl}(A)$ then $B$ is $\pi g N^*$-closed.

**Proof.** Since $A$ is $\pi g N^*$-closed, $N^*(A) \subset U$ whenever $A \subset U$ and $U$ is $\pi$-open. Let $B \subset U$ and $U$ be $\pi$-open. Since $B \subset N^* - \text{cl}(A)$, $N^* - \text{cl}(B) \subset N^* - \text{cl}(A) \subset U$. Hence $B$ is $\pi g N^*$-closed.

2.9. **Theorem.** Let $A$ be a $\pi g N^*$-closed set in $X$. Then $N^* - \text{cl}(A) - A$ does not contain any non empty $\pi$-closed set.

**Proof.** Let $F$ be a non empty $\pi$-closed set such that $F \subset N^* - \text{cl}(A) - A$. Then $F \subset N^* - \text{cl}(A) \cap (X - A) \subset X - A$ implies $A \subset X - F$ where $X - F$ is $\pi$-open. Therefore $N^* - \text{cl}(A) \subset X - F$ implies $F \subset (N^* - \text{cl}(A))^C$. Now $F \subset N^* - \text{cl}(A) (A) \cap (N^* - \text{cl}(A))^C$ implies $F$ is empty.

Reverse implication does not hold.

2.10. **Corollary.** Let $A$ be $\pi g N^*$closed. $A$ is $N^*$-closed iff $N^* - \text{cl}(A) - A$ is $\pi$-closed.

**Proof.** Let $A$ be $N^*$-closed set then $A = N^* - \text{cl}(A)$ implies $N^* - \text{cl}(A) - A = \emptyset$ which is $\pi$-closed.

Conversely if $N^* - \text{cl}(A) - A$ is $\pi$-closed then $A$ is $N^*$-closed.

2.11. **Theorem.** If $A$ is $\pi$-open and $\pi g N^*$-closed. Then $A$ is $N^*$-closed hence clopen.

**Proof.** Let $A$ be regular open. Since $A$ is $\pi g N^*$-closed, $N^* - \text{cl}(A) \subset A$ implies $A$ is $N^*$-closed. Hence $A$ is closed (Since every $\pi$-open, $N^*$-closed set is closed). Therefore $A$ is clopen.

2.12. **Theorem:** For a topological space $X$, the following are equivalent:

(a) $X$ is extremally disconnected.

(b) Every subset of $X$ is $\pi g N^*$-closed.

(c) The topology on $X$ generated by $\pi g N^*$-closed sets.

**Proof.** (a) $\Rightarrow$ (b). Assume $X$ is extremally disconnected. Let $A \subset U$, where $U$ is $\pi$-open in $X$. Since $U$ is $\pi$-open, it is the finite union of regular open sets and $X$ is extremally disconnected, $U$ is finite union of clopen sets and hence $U$ is clopen. Therefore $N^* - \text{cl}(A) \subset \text{cl}(A) \subset \text{cl}(U) \subset U$ implies $A$ is $\pi g N^*$-closed.

(b) $\Rightarrow$ (a). Let $A$ be regular open set of $X$. Since $A$ is $\pi g N^*$-closed by **Theorem 2.11** $A$ is clopen. Hence $X$ is extremally disconnected.

(b) $\Leftrightarrow$ (c) is obvious.
2.13. **Lemma[11]**. If A is a subset of X, then

1. \( X - N^*-\text{cl}(X - A) = N^*-\text{int}(A) \).

2. \( X - N^*-\text{int}(X - A) = N^*-\text{cl}(A) \).

2.14. **Theorem**. A subset A of a topological space X is \( \pi g N^* \)-open iff \( F \subseteq N^*-\text{int}(A) \) whenever \( F \) is \( \pi \)-closed and \( F \subseteq A \).

**Proof.** Let \( F \) be \( \pi \)-closed set such that \( F \subseteq A \). Since \( X - A \) is \( \pi g N^* \)-closed and \( X - A \subseteq X - F \) we have \( F \subseteq N^*-\text{int}(A) \).

Conversely, let \( F \subseteq N^*-\text{int}(A) \) where \( F \) is \( \pi \)-closed and \( F \subseteq A \). Since \( F \subseteq A \) and \( X - F \) is \( \pi \)-open, \( N^*-\text{cl}(X - A) = X - N^*-\text{int}(A) \subseteq X - F \). Therefore \( A \) is \( \pi g N^* \)-open.

2.15. **Theorem**. If \( N^*-\text{int}(A) \subseteq B \subseteq A \) and \( A \) is \( \pi g N^* \)-open then \( B \) is \( \pi g N^* \)-open.

**Proof.** Since \( N^*-\text{int}(A) \subseteq B \subseteq A \) using **Theorem 2.8**, \( N^*-\text{cl}(X - A) \supseteq (X - B) \) implies \( B \) is \( \pi g N^* \)-open.

2.16. **Remark.** For any \( A \subseteq X \), \( N^*-\text{int}(A) \) \( (N^* \text{-cl}(A)) - A \) = \( \phi \).

2.17. **Theorem**. If \( A \subseteq X \) is \( \pi g N^* \)-closed then \( N^* \text{-cl}(A) - A \) is \( \pi g N^* \)-open.

**Proof.** Let \( A \) be \( \pi g N^* \)-closed and \( F \) be a \( \pi \)-closed set such that \( F \subseteq N^* \text{-cl}(A) - A \). By **Theorem 2.9**, \( F = \phi \) implies \( F \subseteq N^*-\text{int}(A) \) \( (N^* \text{-cl}(A) - A) \). By **Theorem 2.14**, \( N^* \text{-cl}(A) - A \) is \( \pi g N^* \)-open.

Converse of the above theorem is not true.

2.18. **Example.** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \). Let \( A = \{b\} \). Then \( A \) is not \( \pi g N^* \)-closed but \( N^*-\text{cl}(A) - A = \{a, b\} \) \( \pi g N^* \)-open.

3. **Quasi \( N^* \)-normal spaces**

3.1. **Definition.** A topological space \( X \) is said to be \( N^* \)-normal (resp. quasi \( N^* \)-normal, mildly \( N^* \)-normal) if for every pair of disjoint closed (resp. \( \pi \)-closed, regularly closed) subsets \( H, K \) of \( X \), there exist disjoint \( N^* \)-open sets \( U, V \) of \( X \) such that \( H \subseteq U \) and \( K \subseteq V \).

3.2. **Example.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\} \). The pair of disjoint closed subsets of \( X \) are \( A = \phi \) and \( B = \{d\} \). Then \( N^* \)-closed sets in \( X \) are \( X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c, d\}, \{b, c\}, \{a, c\}, \{b, d\}, \{a, c, d\}, \{b, c, d\} \). Also \( U = \{b\} \).
and \( V = \{c, d\} \) are \( N^* \)-open sets such that \( A \subset U \) and \( B \subset V \). Hence \( X \) is \( N^* \)-normal but it is not normal.

### 3.3. Example

Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\} \). The pair of disjoint \( \pi \)-closed subsets of \( X \) are \( A = \{a\} \) and \( B = \{c\} \). Also \( U = \{a\} \) and \( V = \{b, c, d\} \) are open sets such that \( A \subset U \) and \( B \subset V \). Hence \( X \) is quasi-normal as well as quasi \( N^* \)-normal because every open set is \( N^* \)-open set.

By the definitions and examples stated above, we have the following diagram:

\[
\text{normality} \implies \text{quasi-normality} \implies \text{mild-normality} \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
\text{\( N^* \)-normality} \implies \text{quasi \( N^* \)-normality} \implies \text{mild \( N^* \)-normality}
\]

### 3.4. Theorem

For topological space \( X \), the following are equivalent:

(a) \( X \) is quasi \( N^* \)-normal.

(b) For any disjoint \( \pi \)-closed sets \( H \) and \( K \), there exist disjoint \( N^*g \)-open sets \( U \) and \( V \) such that \( H \subset U \) and \( K \subset V \).

(c) For any disjoint \( \pi \)-closed sets \( H \) and \( K \), there exist disjoint \( \pi gN^* \)-open sets \( U \) and \( V \) such that \( H \subset U \) and \( K \subset V \).

(d) For any \( \pi \)-closed set \( H \) and any \( \pi \)-open set \( V \) containing \( H \), there exist a \( N^*g \)-open set \( U \) of \( X \) such that \( H \subset U \subset N^*-\text{cl}(U) \subset V \).

(e) For any \( \pi \)-closed set \( H \) and any \( \pi \)-open set \( V \) containing \( H \), there exist a \( \pi gN^* \)-open set \( U \) of \( X \) such that \( H \subset U \subset N^*-\text{cl}(U) \subset V \).

**Proof.** (a) \( \implies \) (b), (b) \( \implies \) (c), (d) \( \implies \) (e), (c) \( \implies \) (d) and (e) \( \implies \) (a).

(a) \( \implies \) (b). Let \( X \) be quasi \( N^* \)-normal. Let \( H, K \) be disjoint \( \pi \)-closed sets of \( X \). By assumption, there exist disjoint \( N^* \)-open sets \( U, V \) such that \( H \subset U \) and \( K \subset V \). Since every \( N^* \)-open set is \( N^*g \)-open, \( U \) and \( V \) are \( N^*g \)-open sets such that \( H \subset U \) and \( K \subset V \).

(b) \( \implies \) (c). Let \( H, K \) be two disjoint \( \pi \)-closed sets. By assumption, there exists \( N^*g \)-open sets \( U \) and \( V \) such that \( H \subset U \) and \( K \subset V \). Since \( N^*g \)-open set is \( \pi gN^* \)-open, \( U \) and \( V \) are \( \pi gN^* \)-open sets such that \( H \subset U \) and \( K \subset V \).
(d) \Rightarrow (e). Let H be any $\pi$-closed set and V be any $\pi$-open set containing H. By assumption, there exist $N^*g$-open set U of X such that $H \subseteq U \subseteq N^*\text{cl}(U) \subseteq V$. Since, every $N^*g$-open set is $\pi gN^*$-open, there exist $\pi gN^*$-open sets U of X such that $H \subseteq U \subseteq N^*\text{cl}(U) \subseteq V$.

(c) \Rightarrow (d). Let H be any $\pi$-closed set and V be any $\pi$-open set containing H. By assumption, there exist $\pi gN^*$-open sets U and W such that $H \subseteq U$ and $X - V \subseteq W$. By Theorem 2.14, we get $X - V \subseteq N^*\text{int}(W)$ and $N^*\text{cl}(U) \cap N^*\text{int}(W) = \emptyset$. Hence $H \subseteq U \subseteq N^*\text{cl}(U) \subseteq X - N^*\text{int}(W) \subseteq V$.

(e) \Rightarrow (a). Let H, K be any two disjoint $\pi$-closed set of X. Then $H \subseteq X - K$ and $X - K$ is $\pi$-open. By assumption, there exist $\pi gN^*$-open set G of X such that $H \subseteq G \subseteq N^*\text{cl}(G) \subseteq X - K$. Put $U = N^*\text{int}(G)$, $V = X - N^*\text{cl}(G)$. Then $U$ and $V$ are disjoint $N^*$-open sets of X such that $H \subseteq U \subseteq N^*\text{cl}(U) \subseteq X - N^*\text{int}(W) \subseteq V$.

4. Some Functions

4.1. Definition. A function $f : X \rightarrow Y$ is said to be

1. almost closed\(^9\) (resp. almost $N^*$-closed, almost $N^*$-g-closed) if $f(F)$ is closed (resp. $N^*$-closed, $N^*g$-closed) in Y for every $F \in \text{RC}(X)$.

2. $\pi gN^*$-closed (resp. almost $\pi gN^*$-closed) if for every closed set (resp. regularly closed) $F$ of $X$, $f(F)$ is $\pi gN^*$-closed in Y.

3. $\pi$-continuous\(^2\) (resp. $\pi g\alpha$-continuous\(^1\), $\pi gN^*$-continuous) if $f^{-1}(F)$ is $\pi$-closed (resp. $\pi g\alpha$-closed, $\pi gN^*$-closed) in $X$ for every closed set $F$ of $Y$.

4. almost continuous\(^9\) (resp. almost $\pi$-continuous\(^2\), almost $\pi g\alpha$-continuous\(^1\), almost $\pi gN^*$-continuous) if $f^{-1}(F)$ is closed (resp. $\pi$-closed, $\pi g\alpha$-closed, $\pi gN^*$-closed) in $X$ for every regularly closed set $F$ of $Y$.

5. rc-preserving\(^7\) if $f(F)$ is regularly closed in $Y$ for every $F \in \text{RC}(X)$.

From the definitions stated above, we obtain the following diagram:
closed $\Rightarrow$ $\alpha$-closed $\Rightarrow$ $\alpha g$-closed $\Rightarrow$ $\pi g \alpha$-closed

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

al.-closed $\Rightarrow$ al.N*-closed $\Rightarrow$ al. N*g-closed $\Rightarrow$ al. $\pi g N*$-closed

where al. = almost

Moreover, by the following examples, we realize that none of the implications is reversible.

4.2. Example. $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\}, \{a, b, d\}$ and $\sigma = \{\phi, \{a\}, \{c, d\}, \{a, c, d\} \}$ $\{d\}, \{a, d\}, X$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function, then $f$ is $\pi g \alpha$-closed as well as $\pi g N*$-closed but not $\pi g$-closed. Since $A = \{c\}$ is not $\pi g$-closed in $(X, \sigma)$.

4.3. Example. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\}, \{a, b, d\}, \{b, d\}, \{b, c, d\}, X\}$ and $\sigma = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{d\}, \{a, d\}\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Then $f$ is almost $\pi g \alpha$-closed as well as almost $\pi g N*$-closed but not $\pi g N*$-closed. Since $A = \{a\}$ is not $\pi g N*$-closed.

4.4. Theorem. If $f : X \rightarrow Y$ is an almost $\pi$-continuous and $\pi g N*$-closed function, then $f(A)$ is $\pi g N*$-closed in $Y$ for every $\pi g N*$-closed set $A$ of $X$.

Proof. Let $A$ be any $\pi g N*$-closed set $A$ of $X$ and $V$ be any $\pi$-open set of $Y$ containing $f(A)$. Since $f$ is almost $\pi$-continuous, $\pi g N*$-closed, $f^{-1}(V)$ is $\pi$-open in $X$ and $A \subseteq f^{-1}(V)$. Therefore $N^*-cl(A) \subseteq f^{-1}(V)$ and hence $f(N^*-cl(A)) \subseteq V$. Since $f$ is $\pi g N*$-closed, $f(N^*-cl(A))$ is $\pi g N*$-closed in $Y$. And hence we obtain $N^*-cl(f(A)) \subseteq N^*-cl(f(N^*-cl(A))) \subseteq V$. Hence $f(A)$ is $\pi g N*$-closed in $Y$.

4.5. Theorem. A surjection $f : X \rightarrow Y$ is almost $\pi g N*$-closed if and only if for each subset $S$ of $Y$ and each $U \in RO(X)$ containing $f^{-1}(S)$ there exists a $\pi g N*$-open set $V$ of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. Necessity. Suppose that $f$ is almost $\pi g N*$-closed. Let $S$ be a subset of $Y$ and $U \in RO(X)$ containing $f^{-1}(S)$. If $V = Y - f(X - U)$, then $V$ is a $\pi g N*$-open set of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Sufficiency. Let $F$ be any regular closed set of $X$. Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F \in RO(X)$. There exists $\pi g N*$-open set $V$ of $Y$ such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, we have $f(F) \subseteq Y - V$ and $F \subseteq X - f^{-1}(V) \subseteq f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ and $f(f)$ is $\pi g N*$-closed in $Y$ which shows that $f$ is almost $\pi g N*$-closed.
5. Preservation Theorem

5.1. Theorem. If \( f : X \rightarrow Y \) is an almost \( \pi g N^* \)-continuous, rc-preserving injection and \( Y \) is quasi \( N^* \)-normal then \( X \) is quasi \( N^* \)-normal.

Proof. Let \( A \) and \( B \) be any disjoint \( \pi \)-closed sets of \( X \). Since \( f \) is an rc-preserving injection, \( f(A) \) and \( f(B) \) are disjoint \( \pi \)-closed sets of \( Y \). Since \( Y \) is quasi \( N^* \)-normal, there exist disjoint \( N^* \)-open sets \( U \) and \( V \) of \( Y \) such that \( f(A) \subseteq U \) and \( f(B) \subseteq V \). Now if \( G = \text{int}(\text{cl}(U)) \) and \( H = \text{int}(\text{cl}(V)) \). Then \( G \) and \( H \) are regularly open sets such that \( f(A) \subseteq G \) and \( f(B) \subseteq H \). Since \( f \) is almost \( \pi g N^* \)-continuous, \( f^{-1}(G) \) and \( f^{-1}(H) \) are disjoint \( \pi g N^* \)-open sets containing \( A \) and \( B \) which shows that \( X \) is quasi \( N^* \)-normal.

5.2. Theorem. If \( f : X \rightarrow Y \) is \( \pi \)-continuous, almost \( N^* \)-closed surjection and \( X \) is quasi \( N^* \)-normal space then \( Y \) is \( N^* \)-normal.

Proof. Let \( A \) and \( B \) be any two disjoint closed sets of \( Y \). Then \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint \( \pi \)-closed sets of \( X \). Since \( X \) is quasi \( N^* \)-normal, there exist disjoint \( N^* \)-open sets of \( U \) and \( V \) such that \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \). Let \( G = \text{int}(\text{cl}(V)) \) and \( H = \text{int}(\text{cl}(V)) \). Then \( G \) and \( H \) are disjoint regularly open sets of \( X \) such that \( f^{-1}(A) \subseteq G \) and \( f^{-1}(B) \subseteq H \). Set \( K = Y - f(X - G) \) and \( L = Y - f (X - H) \). Then \( K \) and \( L \) are \( N^* \)-open sets of \( Y \) such that \( A \subseteq K \), \( B \subseteq L \), \( f^{-1}(K) \subseteq G \), \( f^{-1}(L) \subseteq H \). Since \( G \) and \( H \) are disjoint, \( K \) and \( L \) are disjoint. Since \( K \) and \( L \) are \( N^* \)-open and we obtain \( A \subseteq N^* \text{int}(K) \), \( B \subseteq N^* \text{int}(L) \) and \( N^* \text{int}(K) \cap N^* \text{int}(L) = \phi \). Therefore \( Y \) is \( N^* \)-normal.

5.3. Theorem. Let \( f : X \rightarrow Y \) be an almost \( \pi \)-continuous and almost \( \pi g N^* \)-closed surjection. If \( X \) is quasi \( N^* \)-normal space then \( Y \) is quasi \( N^* \)-normal.

Proof. Let \( A \) and \( B \) be any disjoint \( \pi \)-closed sets of \( Y \). Since \( f \) is almost \( \pi \)-continuous, \( f^{-1}(A) \), \( f^{-1}(B) \) are disjoint closed subsets of \( X \). Since \( X \) is quasi \( N^* \)-normal, there exist disjoint \( N^* \)-open sets \( U \) and \( V \) of \( X \) such that \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \). Put \( G = \text{int}(\text{cl}(U)) \) and \( H = \text{int}(\text{cl}(V)) \). Then \( G \) and \( H \) are disjoint regularly open sets of \( X \) such that \( f^{-1}(A) \subseteq G \) and \( f^{-1}(B) \subseteq H \). By Theorem 4.5, there exist \( \pi g N^* \)-open sets \( K \) and \( L \) of \( Y \) such that \( A \subseteq K \), \( B \subseteq L \), \( f^{-1}(K) \subseteq G \) and \( f^{-1}(L) \subseteq H \). Since \( G \) and \( H \) are disjoint, so are \( K \) and \( L \) by Theorem 2.14, \( A \subseteq N^* \text{int}(K) \), \( B \subseteq N^* \text{int}(L) \) and \( N^* \text{int}(K) \cap N^* \text{int}(L) = \phi \). Therefore, \( Y \) is quasi \( N^* \)-normal.

5.3. Corollary. If \( f : X \rightarrow Y \) is almost continuous and almost closed surjection and \( X \) is a normal space, then \( Y \) is quasi \( N^* \)-normal.

Proof. Since every almost closed function is almost \( \pi g N^* \)-closed so \( Y \) is quasi \( N^* \)-normal.
REFERENCES


