ηg-closed Sets and η-Normal Spaces in Topological Spaces

Hamant Kumar
Department of Mathematics
Veerangana Avantibai Government Degree College, Atrauli-Aligarh, U. P. (India)

Abstract. The aim of this paper is to introduce and study a new class of sets called ηg-closed sets and a new class of spaces called η-normal spaces. The relationships among βg-normal, α-normal, s-normal and η-normal spaces are investigated. Moreover, we introduce the concept of η-generalized closed functions. We also obtain some characterizations and preservation theorems of η-normal spaces, in the forms of generalized η-closed and η-generalized closed functions.

Key Words: η-open, gn-closed, and ηg-closed sets; η-normal spaces; η-closed and η-ηg-closed functions.

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1. Introduction


2. Preliminaries

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and f : (X, τ) → (Y, σ) (or simply f : X → Y) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by cl(A) and int(A), respectively.

2.1 Definition. A subset A of a space X is said to be:

1. (1) regular open [15] if A = int(cl(A)).
   (2) regular closed [15] if A = cl(int(A)).
   (3) s-open [9] if A ⊆ cl(int(A)).
   (4) α-open [13] if A ⊆ int(cl(int(A))).
   (5) η-open [16] if A ⊆ int(cl(int(A))) ∪ cl(int(A)).
The complement of a s-open (resp. α-open, η-open) set is called s-closed (resp. α-closed, η-closed).

The intersection of all s-closed (resp. α-closed, η-closed) sets containing A is called the s-closure (resp. α-closure, η-closure) of A and is denoted by s-cl(A) (resp. α-cl(A), η-cl(A)). The η-interior of A, denoted by η-int(A) is defined to be the union of all η-open sets contained in A.

The family of all η-open (resp. η-closed, regular open, regular closed, s-open, s-closed, α-open, α-closed) sets of a space X is denoted by η-O(X) (resp. η-C(X), R-O(X), R-C(X), S-O(X), S-C(X), α-O(X), α-C(X)).

2.2 Definition. A subset A of a space (X, τ) is said to be
1. g-closed [10] if cl(A) ⊆ U whenever A ⊆ U and U ∈ τ.
3. sg-closed [4] if s-cl(A) ⊆ U whenever A ⊆ U and U ∈ S(O(X)).
5. ga-closed [12] if α-cl(A) ⊆ U whenever A ⊆ U and U ∈ α-O(X).
7. ηg-closed if η-cl(A) ⊆ U whenever A ⊆ U and U ∈ η-O(X).

The complement of g-closed (resp. gs-closed, sg-closed, ag-closed, ga-closed, gn-closed, ηg-closed) set is said to be g-open (resp. gs-open, sg-open, ag-open, ga-open, gn-open, ηg-open).

2.3 Remark. We have the following implications for the properties of subsets:

Where none of the implications is reversible as can be seen from the following examples:

2.4 Example. Let X = {a, b, c, d} and τ = {∅, {b, d}, {a, b, d}, {b, c, d}, X}. Then
1. closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}.
2. g-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.
3. s-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}.
4. gs-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.
5. sg-closed sets in (X, τ) are φ, X, {a}, {c}, {a, b, c}, {a, c, d}.
6. α-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}.
7. ag-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.
8. ga-closed sets in (X, τ) are φ, X, {a}, {c}, {a, d}, {a, b, d}, {a, c, d}.
9. η-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}.
10. gn-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
11. ηg-closed sets in (X, τ) are φ, X, {a}, {c}, {a, c}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

2.5 Example. Let X = {a, b, c, d} and τ = {∅, {a}, {b}, {a, b}, {a, c}, {a, b, c}, X}. Then
1. closed sets in (X, τ) are φ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
2. g-closed sets in (X, τ) are φ, X, {d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
3. s-closed sets in (X, τ) are φ, X, {d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
4. gs-closed sets in (X, τ) are φ, X, {d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
5. sg-closed sets in (X, τ) are φ, X, {d}, {b, d}, {a, b, d}, {a, c, d}, {b, c, d}.
6. α-closed sets in (X, τ) are φ, X, {d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
7. ag-closed sets in (X, τ) are φ, X, {d}, {a, b, d}, {a, c, d}, {b, c, d}.
8. ga-closed sets in (X, τ) are φ, X, {d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(9) η-closed sets in (X, S) are φ, X, {a}, {c}, {d}, {a, c}, {b, d}, {b, d, c, e}, {a, c, d}, {b, d, e}.
(10) gn-closed sets in (X, S) are φ, X, {b}, {c}, {d}, {a, c}, {b, d}, {b, d, c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(11) ng-closed sets in (X, S) are φ, X, {b}, {c}, {d}, {a, c}, {b, d}, {b, d, c, d}, {a, c, d}, {b, c, d}.

3. η-normal Spaces

3.1 Definition. A space X is said to be η-normal if for any pair of disjoint closed sets A and B, there exist disjoint η-open sets U and V such that A ⊂ U and B ⊂ V.

3.2 Definition. A space X is said to be α-normal [3] (resp. s-normal [11], β*-normal [11]) if for any pair of disjoint closed sets A and B, there exist disjoint α-open (resp. s-open, β*-open) sets U and V such that A ⊂ U and B ⊂ V.

3.3 Remark. The following diagram holds for a topological space (X, S):

normal → β*-normal → α-normal → s-normal → η-normal

None of these implications is reversible as shown by the following examples.

3.4 Example. Let X = {a, b, c} and S = {φ, {a}, {b, c}, X}. Then the space (X, S) is normal as well as η-normal.

3.5 Example. Let X = {a, b, c, d} and S = {φ, {a}, {b}, {a, b}, {b, c}, {b, d}, {a, b, d}, {b, c, d}, {a, b, d, c}, {b, c, d}, X}. Let A = {c} and B = {d} be disjoint closed sets, there exist disjoint s-open sets U = {a, c} and V = {b, d} such that A ⊂ U and B ⊂ V. Then the space (X, S) is s-normal as well as η-normal, since every s-open set is η-open. But it is neither normal nor α-normal, because U and V are neither open nor α-open sets.

3.6 Example. Let X = {a, b, c, d} and S = {φ, {a}, {c}, {a, b}, {a, c}, {b, d}, {c, d}, {a, b, d}, {b, c, d}, X}. Let A = {a} and B = {c} be disjoint closed sets, there exist disjoint open sets U = {a} and V = {c} such that A ⊂ U and B ⊂ V. Then the space (X, S) is normal as well as α-normal, s-normal, η-normal, since every open set is α-open, s-open and η-open.

3.7 Example. Let X = {a, b, c, d} and S = {φ, {a}, {c}, {a, b}, {a, c}, {b, d}, X}. Then the space (X, S) is normal as well as α-normal, s-normal, η-normal, since every open set is α-open, s-open and η-open.

3.8 Theorem. For a space X the following are equivalent:
(1) X is η-normal,
(2) For every pair of open sets U and V whose union is X, there exist η-closed sets A and B such that A ⊂ U, B ⊂ V and A ∪ B = X,
(3) For every closed set H and every open set K containing H, there exists an η-open set U such that H ⊂ U ⊂ η-cl(U) ⊂ K.

Proof. (1) ⇒ (2) : Let U and V be a pair of open sets in an η-normal space X such that X = U ∪ V. Then X \ U, X \ V are disjoint closed sets. Since X is η-normal, there exist disjoint η-open sets U1 and V1 such that X \ U ⊂ U1 and X \ V ⊂ V1. Let A = X \ U1, B = X \ V1. Then A and B are η-closed sets such that A ⊂ U, B ⊂ V and A ∪ B = X.

(2) ⇒ (3) : Let H be a closed set and K be an open set containing H. Then X \ H and K are open sets whose union is X. Then by (2), there exist η-closed sets M1 and M2 such that M1 ⊂ X \ H and M2 ⊂ K and M1 ∪ M2 = X. Then H ⊂ X \ M1, X \ K ⊂ X \ M2 and (X \ M1) ∩ (X \ M2) = φ. Let U = X \ M1 and V = X \ M2. Then U and V are disjoint η-open sets such that H ⊂ U ⊂ X \ V ⊂ K. As X \ V is η-closed set, we have η-cl(U) ⊂ X \ V and H ⊂ η-cl(U) ⊂ K.

(3) ⇒ (1) : Let H1 and H2 be any two disjoint closed sets of X. Put K = X \ H2, then H2 ∩ K = φ, H1 ⊂ K, where K is an open set. Then by (3), there exists an η-open set U of X such that H1 ⊂ U ⊂ η-cl(U) ⊂ K. It follows that H2 ⊂ X \ η-cl(U) = V , say, then V is η-open and U ∩ V = φ. Hence H1 and H2 are separated by η-open sets U and V. Therefore X is η-normal.

4. η-normal Spaces with Some Related Functions

4.1 Definition. A function f : X → Y is called
(1) R-map [5] if f \−1(V) is regular open in X for every regular open set V of Y,
(2) completely continuous [1] if f \−1(V) is regular open in X for every open set V of Y,
(3) rc-continuous [6] if for each regular closed set F in Y, f \−1(F) is regular closed in X.
4.2 Definition. A function \( f : X \rightarrow Y \) is called
(1) strongly \( \eta \)-open if \( f(U) \in \eta \)-O(Y) for each \( U \in \eta \)-O(X),
(2) strongly \( \eta \)-closed if \( f(U) \in \eta \)-C(Y) for each \( U \in \eta \)-C(X),
(3) almost \( \eta \)-irresolute if for each \( x \in X \) and each \( \eta \)-neighbourhood \( V \) of \( f(x) \), \( \eta \)-cl(f(\( -1 \)(V))) is a \( \eta \)-neighbourhood of \( x \).

4.3 Theorem. A function \( f : X \rightarrow Y \) is strongly \( \eta \)-closed if and only if for each subset \( A \) in \( Y \) and for each \( \eta \)-open set \( U \) in \( X \) containing \( f^{-1}(A) \), there exists an \( \eta \)-open set \( V \) containing \( A \) such that \( f^{-1}(U) \subset V \).

Proof. \((\Rightarrow)\) : Suppose that \( f \) is strongly \( \eta \)-closed. Let \( A \) be a subset of \( Y \) and \( U \in \eta \)-O(X) containing \( f^{-1}(A) \). Put \( V = Y - f(X - U) \), then \( V \) is an \( \eta \)-open set of \( Y \) such that \( A \subset V \) and \( f^{-1}(V) \subset U \).

\((\Leftarrow)\) : Let \( K \) be any \( \eta \)-closed set of \( X \). Then \( f^{-1}(Y - f(K)) \subset X - K \) and \( X - K \in \eta \)-O(X). There exists an \( \eta \)-open set \( V \) of \( Y \) such that \( Y - f(K) \subset V \) and \( f^{-1}(V) \subset X - K \). Therefore, we have \( f(K) \supset Y - V \) and \( K \subset f^{-1}(Y - V) \). Hence, we obtain \( f(K) = Y - V \) and \( f(K) \) is \( \eta \)-closed in \( Y \). This shows that \( f \) is strongly \( \eta \)-closed.

4.4 Lemma. For a function \( f : X \rightarrow Y \), the following are equivalent:
(1) \( f \) is almost \( \eta \)-irresolute,
(2) \( f^{-1}(V) \subset \eta \)-int(\( \eta \)-cl(f(\( -1 \)(V)))) for every \( V \in \eta \)-O(Y).

4.5 Theorem. A function \( f : X \rightarrow Y \) is almost \( \eta \)-irresolute if and only if \( f(\eta \)-cl(U)) \( \subset \eta \)-cl(f(U)) for every \( U \in \eta \)-O(X).

Proof. \((\Rightarrow)\) : Let \( U \in \eta \)-O(X). Suppose \( y \in \eta \)-cl(f(U)). Then there exists \( V \in \eta \)-O(Y) such that \( V \cap f(U) = \phi \). Hence, \( f^{-1}(V) \cap U = \phi \). Since \( U \in \eta \)-O(X), we have \( \eta \)-int(\( \eta \)-cl(f(\( -1 \)(V)))) \( \cap \eta \)-cl(U) = \( \phi \). Then by Lemma 4.4, \( f^{-1}(V) \cap \eta \)-cl(U) = \( \phi \) and hence \( V \cap f(\eta \)-cl(U)) = \( \phi \). This implies that \( y \not\in f(\eta \)-cl(U)).

\((\Leftarrow)\) : If \( V \in \eta \)-O(Y), then \( M = X - \eta \)-cl(f(\( -1 \)(V))) \( \in \eta \)-O(X). By hypothesis, \( f(\eta \)-cl(M)) \( \subset \eta \)-cl(f(M)) and hence \( X - \eta \)-int(\( \eta \)-cl(f(\( -1 \)(V)))) = \( \eta \)-cl(M) \( \subset f^{-1}(\eta \)-cl(f(M))) \( \subset f^{-1}(\eta \)-cl(f(f(\( -1 \)(V)))) \( \subset f^{-1}(\eta \)-cl(Y - V)) = f^{-1}(Y - V) = X - f^{-1}(V) \). Therefore, \( f^{-1}(V) \subset \eta \)-int(\( \eta \)-cl(f(\( -1 \)(V)))) by Lemma 4.4, \( f \) is almost \( \eta \)-irresolute.

4.6 Theorem. If \( f : X \rightarrow Y \) is a strongly \( \eta \)-open continuous almost \( \eta \)-irresolute function from a \( \eta \)-normal space \( X \) onto a space \( Y \), then \( Y \) is \( \eta \)-normal.

Proof. Let \( A \) be a closed subset of \( Y \) and \( B \) be an open set containing \( A \). Then by continuity of \( f \), \( f^{-1}(A) \) is closed and \( f^{-1}(B) \) is an open set of \( X \) such that \( f^{-1}(A) \subset f^{-1}(B) \). As \( X \) is \( \eta \)-normal, there exists an \( \eta \)-open set \( U \) in \( X \) such that \( f^{-1}(A) \subset U \subset \eta \)-cl(U) \( \subset f^{-1}(B) \) by Theorem 3.8. Then, \( f(f^{-1}(A)) \subset f(U) \subset f(\eta \)-cl(U)) \( \subset f(f^{-1}(B)) \). Since \( f \) is strongly \( \eta \)-open almost \( \eta \)-irresolute surjection, we obtain \( A \subset f(U) \subset \eta \)-cl(f(U)) \( \subset B \) and then by Theorem 3.8, the space \( Y \) is \( \eta \)-normal.

4.7 Theorem. If \( f : X \rightarrow Y \) is an \( \eta \)-closed continuous function from an \( \eta \)-normal space \( X \) onto a space \( Y \), then \( Y \) is \( \eta \)-normal.

Proof. Let \( M_1 \) and \( M_2 \) be disjoint closed sets. Then \( f^{-1}(M_1) \) and \( f^{-1}(M_2) \) are closed sets. Since \( X \) is \( \eta \)-normal, then there exist disjoint \( \eta \)-open sets \( U \) and \( V \) such that \( f^{-1}(M_1) \subset U \) and \( f^{-1}(M_2) \subset V \). By Theorem 4.3, there exist \( \eta \)-open sets \( A \) and \( B \) such that \( M_1 \subset A \), \( M_2 \subset B \), \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). Also, \( A \) and \( B \) are disjoint. Thus, \( Y \) is \( \eta \)-normal.

5. \( \eta \)-generalized Closed Functions

5.1 Definition. A function \( f : X \rightarrow Y \) is said to be
(1) \( \eta \)-closed [8] if \( f(A) \) is \( \eta \)-closed in \( Y \) for each closed set \( A \) of \( X \),
(2) \( \eta \)-ng-closed if \( f(A) \) is \( \eta \)-ng-closed in \( Y \) for each closed set \( A \) of \( X \),
(3) \( \eta \)-g\( \eta \)-closed [18] if \( f(A) \) is \( \eta \)-g\( \eta \)-closed in \( Y \) for each closed set \( A \) of \( X \).

5.2 Definition. A function \( f : X \rightarrow Y \) is said to be
(1) quasi \( \eta \)-closed if \( f(A) \) is closed in \( Y \) for each \( A \in \eta \)-C(X),
(2) \( \eta \)-ng-closed if \( f(A) \) is \( \eta \)-ng-closed in \( Y \) for each \( A \in \eta \)-C(X),
(3) \( \eta \)-ng\( \eta \)-closed if \( f(A) \) is \( \eta \)-ng\( \eta \)-closed in \( Y \) for each \( A \in \eta \)-C(X),
(4) almost \( \eta \)-g\( \eta \)-closed if \( f(A) \) is \( \eta \)-g\( \eta \)-closed in \( Y \) for each \( A \in R-C(X) \).

5.3 Definition. A function \( f : X \rightarrow Y \) is said to be \( \eta \)-g\( \eta \)-continuous if \( f^{-1}(K) \) is \( \eta \)-g\( \eta \)-closed in \( X \) for every \( K \in \eta \)-C(Y).

5.4 Definition. A function \( f : X \rightarrow Y \) is said to be \( \eta \)-irresolute [8] if \( f^{-1}(V) \in \eta \)-O(X) for every \( V \in \eta \)-O(Y).

5.5 Theorem. Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be functions. Then
(1) the composition $gof : X \rightarrow Z$ is $\eta$-$g\eta$-closed if $f$ is $\eta$-$g\eta$-closed and $g$ is continuous $\eta$-$g\eta$-closed.
(2) the composition $gof : X \rightarrow Z$ is $\eta$-$g\eta$-closed if $f$ is strongly $\eta$-closed and $g$ is $\eta$-$g\eta$-closed.
(3) the composition $gof : X \rightarrow Z$ is $\eta$-$g\eta$-closed if $f$ is quasi $\eta$-closed and $g$ is $g\eta$-closed.

5.6 Theorem. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let the composition $gof : X \rightarrow Z$ be $\eta$-$g\eta$-closed. If $f$ is a $\eta$-irresolute surjection, then $g$ is $\eta$-$g\eta$-closed.

Proof. Let $K \in \eta$-$C(Y)$. Since $f$ is $\eta$-irresolute and surjective, $f^{-1}(K) \in \eta$-$C(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Hence, $g(K)$ is $g\eta$-closed in $Z$ and hence $g$ is $\eta$-$g\eta$-closed.

5.7 Lemma. A function $f : X \rightarrow Y$ is $\eta$-$g\eta$-closed if and only if for each subset $B$ of $Y$ and each $U \in \eta$-$O(X)$ containing $f^{-1}(B)$, there exists a $g\eta$-open set $V$ of $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. ($\Rightarrow$): Suppose that $f$ is $\eta$-$g\eta$-closed. Let $B$ be a subset of $Y$ and $U \in \eta$-$O(X)$ containing $f^{-1}(B)$. Put $V = Y - f(X - U)$, then $V$ is a $g\eta$-open set of $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

($\Leftarrow$): Let $K$ be any $\eta$-closed set of $X$. Then $f^{-1}(Y - f(K)) \subseteq X - K$ and $X - K \in \eta$-$O(X)$. There exists a $g\eta$-open set $V$ of $Y$ such that $Y - f(K) \subseteq V$ and $f^{-1}(V) \subseteq X - K$. Therefore, we have $f(K) \supseteq Y - V$ and $K \subseteq f^{-1}(Y - V)$. Hence, we obtain $f(K) = Y - V$ and $f(K)$ is $g\eta$-closed in $Y$. This shows that $f$ is $\eta$-$g\eta$-closed.

5.8 Theorem. If $f : X \rightarrow Y$ is continuous $\eta$-$g\eta$-closed, then $f(H)$ is $g\eta$-closed in $Y$ for each $g\eta$-closed set $H$ of $X$.

Proof. Let $H$ be any $g\eta$-closed set of $X$ and $V$ an open set of $Y$ containing $f(H)$. Since $f^{-1}(V)$ is an open set of $X$ containing $H$, $\eta$-$cl(H) \subseteq f^{-1}(V)$ and hence $f(\eta$-$cl(H)) \subseteq V$. Since $f$ is $\eta$-$g\eta$-closed and $\eta$-$cl(H) \in \eta$-$C(X)$, we have $\eta$-$cl(f(H)) \subseteq \eta$-$cl(f(\eta$-$cl(H))) \subseteq V$. Therefore, $f(H)$ is $g\eta$-closed in $Y$.

5.9 Remark. Every $\eta$-irresolute function is $\eta$-$g\eta$-continuous but not conversely.

5.10 Theorem. A function $f : X \rightarrow Y$ is $\eta$-$g\eta$-continuous if and only if $f^{-1}(V)$ is $g\eta$-open in $X$ for every $V \in \eta$-$O(Y)$.

5.11 Theorem. If $f : X \rightarrow Y$ is closed $\eta$-$g\eta$-continuous, then $f^{-1}(K)$ is $g\eta$-closed in $X$ for each $g\eta$-closed set $K$ of $Y$.

Proof. Let $K$ be a $g\eta$-closed set of $Y$ and $U$ an open set of $X$ containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then $V$ is open in $Y$, $K \subseteq V$, and $f^{-1}(V) \subseteq U$. Therefore, we have $\eta$-$cl(K) \subseteq V$ and hence $f^{-1}(\eta$-$cl(K)) \subseteq f^{-1}(V) \subseteq U$. Since $f$ is $\eta$-$g\eta$-continuous, $f^{-1}(\eta$-$cl(K))$ is $g\eta$-closed in $X$ and hence $\eta$-$cl(f^{-1}(K)) \subseteq \eta$-$cl(f^{-1}(\eta$-$cl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $g\eta$-closed in $X$.

5.12 Corollary. If $f : X \rightarrow Y$ is closed $\eta$-irresolute, then $f^{-1}(K)$ is $g\eta$-closed in $X$ for each $g\eta$-closed set $K$ of $Y$.

5.13 Theorem. If $f : X \rightarrow Y$ is an open $\eta$-$g\eta$-continuous bijection, then $f^{-1}(K)$ is $g\eta$-closed in $X$ for every $g\eta$-closed set $K$ of $Y$.

Proof. Let $K$ be a $g\eta$-closed set of $Y$ and $U$ an open set of $X$ containing $f^{-1}(K)$. Since $f$ is an open surjective, $K = f(f^{-1}(K)) \subseteq f(U)$ and $f(U)$ is open. Therefore, $\eta$-$cl(K) \subseteq f(U)$. Since $f$ is injective, $f^{-1}(K) = f^{-1}(\eta$-$cl(K)) \subseteq f^{-1}(f(U)) = U$. Since $f$ is $\eta$-$g\eta$-continuous, $f^{-1}(\eta$-$cl(K))$ is $g\eta$-closed in $X$ and hence $\eta$-$cl(f^{-1}(K)) \subseteq \eta$-$cl(f^{-1}(\eta$-$cl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $g\eta$-closed in $X$.

5.14 Theorem. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let the composition $gof : X \rightarrow Z$ be $\eta$-$g\eta$-closed. If $g$ is an open $\eta$-$g\eta$-continuous bijection, then $f$ is $\eta$-$g\eta$-closed.

Proof. Let $H \in \eta$-$C(X)$. Then $(gof)(H) = g\eta$-closed in $Z$ and $g^{-1}((gof)(H)) = f(H)$. By Theorem 5.13, $f(H)$ is $g\eta$-closed in $Y$ and hence $f$ is $\eta$-$g\eta$-closed.

5.15 Theorem. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let the composition $gof : X \rightarrow Z$ be $\eta$-$g\eta$-closed. If $g$ is a closed $\eta$-$g\eta$-continuous injection, then $f$ is $\eta$-$g\eta$-closed.

Proof. Let $H \in \eta$-$C(X)$. Then $(gof)(H) = g\eta$-closed in $Z$ and $g^{-1}((gof)(H)) = f(H)$. By Theorem 5.11, $f(H)$ is $g\eta$-closed in $Y$ and hence $f$ is $\eta$-$g\eta$-closed.
6. Characterizations of $\eta$-normal Spaces and Some Preservation Theorems

### 6.1 Theorem
For a topological space $X$, the following are equivalent:

(a) $X$ is $\eta$-normal,
(b) for any pair of disjoint closed sets $A$ and $B$ of $X$, there exist disjoint $\eta$-open sets $U$ and $V$ of $X$ such that $A \subset U$ and $B \subset V$,
(c) for each closed set $A$ and each open set $B$ containing $A$, there exists a $\eta$-open set $U$ such that $cl(A) \subset U \subset \eta-cl(U) \subset B$,
(d) for each closed $A$ and each $\eta$-open set $B$ containing $A$, there exists an $\eta$-open set $U$ such that $A \subset U \subset \eta-cl(U) \subset int(B)$,
(e) for each closed $A$ and each $\eta$-open set $B$ containing $A$, there exists a $\eta$-open set $G$ such that $A \subset G \subset \eta-cl(G) \subset int(B)$,
(f) for each closed set $A$ and each open set $B$ containing $A$, there exists an $\eta$-open set $U$ such that $cl(A) \subset U \subset \eta-cl(U) \subset B$,
(g) for each closed set $A$ and each open set $B$ containing $A$, there exists a $\eta$-open set $G$ such that $cl(A) \subset G \subset \eta-cl(G) \subset B$.

**Proof.**
(a) $\iff$ (b) $\iff$ (c): Since every $\eta$-open set is $\eta$-open, it is obvious.

(d) $\implies$ (e) $\implies$ (f): Since every closed (resp. open) set is $g$-closed (resp. $g$-open), it is obvious.

(c) $\implies$ (g): Let $A$ be a closed subset of $X$ and $B$ be an open set such that $A \subset B$. Since $B$ is $g$-open and $A$ is closed, $A \subset int(A)$. Then, there exists an $\eta$-open set $U$ such that $A \subset U \subset \eta-cl(U) \subset int(B)$.

(e) $\implies$ (d): Let $A$ be any closed subset of $X$ and $B$ be a $g$-open set containing $A$. Then there exists a $\eta$-open set $G$ such that $A \subset G \subset \eta-cl(G) \subset int(B)$. Since $G$ is $\eta$-open, $A \subset \eta-int(G)$. Put $U = \eta-int(G)$, then $U$ is $\eta$-open and $A \subset U \subset \eta-cl(U) \subset int(B)$.

(c) $\implies$ (g): Let $A$ be any $g$-closed subset of $X$ and $B$ be an open set such that $A \subset B$. Then $cl(A) \subset B$. Therefore, there exists an $\eta$-open set $U$ such that $cl(A) \subset U \subset \eta-cl(U) \subset B$.

(g) $\implies$ (f): Let $A$ be any $g$-closed subset of $X$ and $B$ be an open set containing $A$. Then there exists a $\eta$-open set $G$ such that $cl(A) \subset G \subset \eta-cl(G) \subset B$. Since $G$ is $\eta$-open and $cl(A) \subset G$, we have $cl(A) \subset \eta-int(G)$, put $U = \eta-int(G)$, then $U$ is $\eta$-open and $cl(A) \subset U \subset \eta-cl(U) \subset B$.

### 6.2 Theorem
If $f : X \to Y$ is a continuous quasi-$\eta$-closed surjection and $X$ is $\eta$-normal, then $Y$ is normal.

**Proof.** Let $M_1$ and $M_2$ be any disjoint closed sets of $Y$. Since $f$ is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of $X$. Since $X$ is $\eta$-normal, there exist disjoint $U_1, U_2 \in \eta-O(X)$ such that $f^{-1}(M_i) \subset U_i$ for $i = 1, 2$. Let $V_i = Y - f(X - U_i)$, then $V_i$ is open in $Y$, $M_i \subset V_i$, and $f^{-1}(V_i) \subset U_i$ for $i = 1, 2$. Since $U_1 \cap U_2 = \emptyset$ and $f$ is surjective, we have $V_1 \cap V_2 = \emptyset$. This shows that $Y$ is normal.

### 6.3 Lemma [17]
A subset $A$ of a space $X$ is $g\eta$-open if and only if $F \cap \eta-int(A)$ whenever $F$ is closed and $F \subset A$.

### 6.4 Theorem
Let $f : X \to Y$ be a closed $\eta$-$g\eta$-continuous injection. If $Y$ is $\eta$-normal, then $X$ is $\eta$-normal.

**Proof.** Let $N_1$ and $N_2$ be disjoint closed sets of $X$. Since $f$ is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of $Y$. By the $\eta$-normality of $Y$, there exist disjoint $V_1, V_2 \in \eta-O(Y)$ such that $f^{-1}(N_i) \subset V_i$ for $i = 1, 2$. Since $f$ is $\eta$-$g\eta$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $\eta$-open sets of $X$ and $N_i \subset f^{-1}(V_i)$ for $i = 1, 2$. Now, put $U_i = \eta-int(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in \eta-O(X)$, $N_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. This shows that $X$ is $\eta$-normal.

### 6.5 Corollary
If $f : X \to Y$ is a closed $\eta$-irresolute injection and $Y$ is $\eta$-normal, then $X$ is $\eta$-normal.

**Proof.** This is an immediate consequence since every $\eta$-irresolute function is $-g\eta$-continuous.

### 6.6 Lemma
A function $f : X \to Y$ is almost $g\eta$-closed if and only if for each subset $B$ of $Y$ and each $U \in \text{R-O}(X)$ containing $f^{-1}(B)$, there exists a $g\eta$-open set $V$ of $Y$ such that $B \subset V$ and $f^{-1}(V) \subset U$.

### 6.7 Lemma
If $f : X \to Y$ is almost $g\eta$-closed, then for each closed set $M$ of $Y$ and each $U \in \text{R-O}(X)$ containing $f^{-1}(M)$, there exists $V \in \eta-O(Y)$ such that $M \subset V$ and $f^{-1}(V) \subset U$.

### 6.8 Theorem
Let $f : X \to Y$ be a continuous almost $g\eta$-closed surjection. If $X$ is normal, then $Y$ is $\eta$-normal.

**Proof.** Let $M_1$ and $M_2$ be any disjoint, closed sets of $Y$. Since $f$ is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of $X$. By the normality of $X$, there exist disjoint open sets $U_1$ and $U_2$ such that $f^{-1}(M_i) \subset U_i$, where $i = 1, 2$. Now, put $G_i = int(cl(U_i))$ for $i = 1, 2$, then $G_i \in \text{R-O}(X)$, $f^{-1}(M_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \emptyset$. By Lemma 6.7, there exists $V_i \in \eta-O(Y)$ such that $M_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where $i = 1, 2$. Since $G_1 \cap G_2 = \emptyset$ and $f$ is surjective, we have $V_1 \cap V_2 = \emptyset$. This shows that $Y$ is $\eta$-normal.

### 6.9 Corollary
If $f : X \to Y$ is a continuous $\eta$-closed surjection and $X$ is normal, then $Y$ is $\eta$-normal.