A STUDY OF UNIFIED DOUBLE INTEGRALS AND LAPLACE TRANSFORMS INVOLVING THE PRODUCT OF GENERAL POLYNOMIALS AND H-FUNCTIONS OF TWO VARIABLES

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ABSTRACT: Double integral evaluated here involve the exponential function the product of two general polynomials and H-Function of two variables. This double integral is unified, useful and most general in nature and capable of yielding a large number of integral double Laplace transforms as their special cases.

INTRODUCTION

The H-Function of two variables [(5), Srivastava et al . 1982, p.82] define in following manner.

$$\mathrm{H}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathrm{H}[z_1, z_2] = \mathrm{H}^{o,n:m_1,n_1,m_2 n_2}_{p,q,:\, p_1,q_1,p_2,q_2}\left[\begin{matrix} z_1 \\ \\ z_2 \end{matrix}\right|$$

$$\left.\begin{matrix} \left(a_j, \alpha_j^{(1)}, \alpha_j^{(2)}\right)_{1,p} : \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,p_1} ; \left(c_j^{(2)}, \gamma_j^{(2)}\right)_{1,p_2} \\ \left(b_j, \beta_j^{(1)}, \beta_j^{(2)}\right)_{1,q} : \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,q_1} ; \left(d_j^{(2)}, \delta_j^{(2)}\right)_{1,q_2} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi_1)\phi_2(\xi_2)\psi(\xi_1, \xi_2)\; z_1^{\xi_1} z_2^{\xi_2} d\xi_1 d\xi_2 \qquad (1.1)$$

The convergence conditions of integral given by (1.1) and other details of thetwo variable H-function can be seen in the book by Srivastave et al. [(5), p. 82, 83]

Srivastava [(3), p.1,equn.(1)] has introduced the general class of polynomials.

$$S_N^M[x] = \sum_{k=0}^{[N/M]} \frac{(-N)_{Mk} A_{N,k} x^k}{k!}, \text{ (N=0,1,2,…..)} \qquad (1.2)$$

When M is an arbitrary positive integer and coefficients $A_{N,k}(N, k \geq 0)$ are arbitrary constants real or complex . On suitability specializing the coefficients $A_{N,k}$ $S_N^M[x]$ yields a number of know polynomials as it special cases. These include, among others, the Jacobi polynomials, the Laguerre polynomials, the Harmite polynomials and several others (Srivastava and Singh [4,pp.158-161]

The double laplace transform occurring herein will be defined and represented in the following manner.

$$L\{f(x_1, x_2); s_1, s_2\} = \int_0^\infty \int_0^\infty e^{-s_1 x_1 - s_2 x_2} f(x_1, x_2) dx_1 dx_2 \qquad (1.3)$$

Main Integral:

$$\int_0^\infty\int_0^\infty (\lambda'_1x_1+\lambda'_2x_2)^{\sigma_1-1}(\lambda''_1x_1+\lambda''_2x_2)^{\sigma_2-1}\exp\left[-s_1(\lambda'_1x_1+\lambda'_2x_2)-s_2(\lambda''_1x_1+\lambda''_2x_2)\right]$$

$$S_{N_1}^{M_1}[C(\lambda'_1x_1+\lambda'_2x_2)^{\rho_1}]\ S_{N_2}^{M_2}[D(\lambda''_1x_1+\lambda''_2x_2)^{\rho_2}]\ H^{o,n:m_1,n_1,m_2n_2}_{p,q,:\,p_1,q_1,p_2,q_2}\left[\begin{matrix} z_1(\lambda'_1x_1+\lambda'_2x_2)^{v_1} \\ z_2(\lambda''_1x_1+\lambda''_2x_2)^{v_2}\end{matrix}\right|$$

$$\left.\begin{matrix}(a_j,\alpha_j^{(1)},\alpha_j^{(2)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};(c_j^{(2)},\gamma_j^{(2)})_{1,p_2}\\ (b_j,\beta_j^{(1)},\beta_j^{(2)})_{1,q}:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};(d_j^{(2)},\delta_j^{(2)})_{1,q_2}\end{matrix}\right]dx_1dx_2$$

$$=\frac{1}{k}\sum_{k_1=0}^{[N_1/M_1]}\sum_{k_2=0}^{[N_2/M_2]}\frac{(-N_1)_{M_{1k_1}}(-N_2)_{M_{2k_2}}}{k_1!\ k_2!\ s_1^{(\sigma_1+\rho_1k_1)}s_2^{(\sigma_2+\rho_2k_2)}}$$

$$H^{o,n\,:\,m_1,n_1+1;\,m_2n_2+1}_{p,q,:\,p_1+1,q,\ \ ;p_2+1,q_2}\left[\begin{matrix}z_1s_1^{-v_1}\\ z_2s_2^{-v_2}\end{matrix}\right|\begin{matrix}(a_j,\alpha_j^{(1)},\alpha_j^{(2)})_{1,p}:\\ (b_j,\beta_j^{(1)},\beta_j^{(2)})_{1,q}:\end{matrix}$$

$$\left.\begin{matrix}(1-\sigma_1-\rho_1k_1,v_1)(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};(1-\sigma_2-\rho_2k_2,v_2)(c_j^{(2)},\gamma_j^{(2)})_{1,p_2}\\ (d_j^{(1)},\delta_j^{(1)})_{1,q_1};\qquad (d_j^{(2)},\delta_j^{(2)})_{1,q_2}\end{matrix}\right]dx_1dx_2$$

Where $k=\begin{vmatrix}\lambda'_1 & \lambda''_1\\ \lambda'_2 & \lambda''_2\end{vmatrix}\neq 0, v_i>0, \text{Re}(s_i)>=0, \left[\text{Re}(\sigma_i)+v_i\min_{1\le j\le m_i}\text{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$, i=1,2

Prof of (2.1) : To Prove (2.1) we first express the H- function of two variables occurring in the left hand side of (2.1) in term of Mellin-Barnes type of contour integrals then interchange the order of ξ_1,ξ_2 and x_1,x_2 integrals we get in following result after little simplification.

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_s}\psi(\xi_1,\xi_2)\phi_1(\xi_1)\phi_2(\xi_2)\,z_1^{\xi_1}z_2^{\xi_2}d\xi_1d\xi_2\,\boldsymbol{\Delta} \tag{2.2}$$

Where in (2.2)

$$\boldsymbol{\Delta}=\int_0^\infty\int_0^\infty(\lambda'_1x_1+\lambda'_2x_2)^{v_1\xi_1+\sigma_1-1}(\lambda''_1x_1+\lambda''_2x_2)^{v_2\xi_2+\sigma_2-1}\exp\left[-s_1(\lambda'_1x_1+\lambda'_2x_2)-s_2(\lambda''_1x_1+\lambda''_2x_2)\right]$$

$$S_{N_1}^{M_1}[C(\lambda'_1x_1+\lambda'_2x_2)^{\rho_1}]S_{N_2}^{M_2}[D(\lambda''_1x_1+\lambda''_2x_2)^{\rho_2}]dx_1dx_2 \tag{2.3}$$

Now we evaluate Δ in following manner:
We have [Wider 1989, p. 241, eqn. (7)]

$$\int_0^\infty\int_0^\infty F(\lambda'_1x_1+\lambda'_2x_2,\lambda''_1x_1+\lambda''_2x_2)dx_1dx_2=\frac{1}{k}\int_0^\infty\int_0^\infty F(u_1,u_2)\,du_1du_2 \tag{2.4}$$

Where k stands for the expression mentioned in (2.1)
If we take $F(\lambda'_1x_1+\lambda'_2x_2,\lambda''_1x_1+\lambda''_2x_2)=f_1(\lambda'_1x_1+\lambda'_2)f_2(\lambda''_1x_1+\lambda''_2x_2)$
Then (2.4) Transformed to

$$\int_0^\infty\int_0^\infty f_1(\lambda'_1x_1+\lambda'_2x_2,)f_2(\lambda''_1x_1+\lambda''_2x_2)dx_1dx_2=\frac{1}{k}\int_0^\infty f_1(u_1)du_1\int_0^\infty f_2(u_2)du_2 \tag{2.5}$$

Consider $f_1(\lambda'_1x_1+\lambda'_2x_2)=(\lambda'_1x_1+\lambda'_2x_2)^{v_1\xi_1+\sigma_1-1}\exp[-s_1(\lambda'_1x_1+\lambda'_2x_2)]\,S_{N_1}^{M_1}[C(\lambda'_1x_1+\lambda'_2x_2)^{\rho_1}]$

$f_2(\lambda''_1x_1+\lambda''_2x_2)=(\lambda''_1x_1+\lambda''_2x_2)^{v_2\xi_2+\sigma_2-1}\exp[-s_2(\lambda'_1x_1+\lambda'_2x_2)]\,S_{N_2}^{M_2}[D(\lambda'_1x_1+\lambda'_2x_2)^{\rho_2}]$

Then form (2.5) we get

$$\int_0^\infty \int_0^\infty (\lambda'_1 x_1 + \lambda'_2 x_2)^{v_1\xi_1+\sigma_1-1} (\lambda''_1 x_1 + \lambda''_2 x_2)^{v_2\xi_2+\sigma_2-1} \exp[-s_1(\lambda'_1 x_1 + \lambda'_2 x_2) - s_2(\lambda'_1 x_1 + \lambda'_2 x_2)]$$

$$S_{N_1}^{M_1}[C(\lambda'_1 x_1 + \lambda'_2 x_2)^{\rho_1}] S_{N_2}^{M_2}[D(\lambda''_1 x_1 + \lambda''_2 x_2)^{\rho_2}] dx_1 dx_2$$

$$= \frac{1}{k} \int_0^\infty u_1^{v_1\xi_1+\sigma_1-1} e^{-s_1 u_1} S_{N_1}^{M_1}[C u_1^{\rho_1}] du_1 \int_0^\infty u_2^{v_2\xi_2+\sigma_2-1} e^{-s_2 u_2} S_{N_2}^{M_2}[D u_2^{\rho_2}] du_2 \quad (2.6)$$

On expressing the general class of polynomials occurring on the right hand side of (2.6) in terms of series with the help of (1.2) interchanging the order of integrals and summation in the result thus obtained and interchanging the u_1 and u_2 integrals .with the help of know formula Gradshteyn and Ryzhik[(2), p. 317, eqn. (3.38), (14)] eqn. (2.2) takes the Following form :

$$\frac{1}{k} \sum_{k_1=0}^{[N_1/M_1]} \sum_{k_2=0}^{[N_2/M_2]} \frac{(-N_1)_{M_1 k_1} (-N_2)_{M_2 k_2}}{k_1!\ k_2!}$$

$$\left\{\frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi_1)\phi_2(\xi_2)\psi(\xi_1,\xi_2) \frac{\Gamma(\sigma_1+\rho_1 k_1+v_1\xi_1) \times \Gamma(\sigma_2+\rho_2 k_2+v_2\xi_2)}{s_1^{\sigma_1+\rho_1 k_1+v_1\xi_1} s_2^{\sigma_2+\rho_2 k_2+v_2\xi_2}} z_1^{\xi_1} z_2^{\xi_2} d\xi_1 d\xi_2\right\} \quad (2.7)$$

Finally on reinterpreting the Mellin-Barnes integral occurring in right hand side of (2.7) in termsH- function of two variables we get the desired results (2.1).

SPECIAL CASES: On account of the usefulness of double Laplace transform define by (1.3) we shall express the result given by (2.1) in the form of double Laplace transform. Thus if we take $\lambda'_1 = \lambda''_2 = 1$, $\lambda''_1 = \lambda'_2 = 0$, $C = 1$, $D = 1$, $\rho_1 = \rho_2 = 1$ in (2.1) we get the following Interesting double Laplace Transform which involve the product of two general class of polynomials and H- function of two variables.

$$L\{x_1^{\sigma_1-1} x_2^{\sigma_2-1}\ S_{N_1}^{M_1}[x_1] S_{N_2}^{M_2}[x_2]\ H\begin{bmatrix} z_1 x_1^{v_1} \\ z_2 x_2^{v_2}\end{bmatrix}; s_1, s_2\}$$

$$= \sum_{k_1=0}^{[N_1/M_1]} \sum_{k_2=0}^{[N_2/M_2]} \frac{(-N_1)_{M_1 k_1} (-N_2)_{M_2 k_2} A_{N_1,k_1} A_{N_1,k_2}}{k_1!\ k_2!\ s_1^{(\sigma_1+k_1)} s_2^{(\sigma_2+k_2)}} H^{**} \quad (3.1)$$

Where H^* occurring in (3.1) stands for the same two variableH- function which occurs on the right hand side of (2.1).

If we put in (2.1) $M_1 = M_2 = s_1 = s_2 = 1, \rho_1 = \rho_2 = C = D = 1$ and $v_1 = v_2 = 0$ and replace A_{N_1,k_1} by $\binom{N_1+\alpha_1}{N_1}$ and A_{N_2,k_2} by $\binom{N_2+\alpha_2}{N_2}$ respectively , then $S_{N_1}^{M_1}, S_{N_2}^{M_2}$occurring therein reduce to Laguerre polynomials Srivastava and Singh [(4), p.159, eqn (1.8)] and reduces the two variable H-function to unity Srivastava at al. [(5), p. 82] we arrive at a known result Dhawan [1, p. 417, eqn (2.2)] after a little simplification.

REFERENCES:

1. Dhawan, G.K. (1968) :Proc . Camb. Phil Soc. 64, 417
2. Gradshteyn , I.S. and Ryzhik, I. M. (1980) : Table of Integrals Series and products. Academic Press, Inc. New York.
3. Srivastava, H.M.(1972) : Indian J. Math 14.1-6
4. Srivastava, H.M. and Singh N.P. (1983) :Rend Circ. mat. Palermo (2) 32, 157-187.
5. Srivastava, H.M, Gupta, K. C. and Goyal, S. P. (1982) The H-Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi
6. Widder, D.V. (1989) : Advanced Calculus, Dover Publications, Inc., New York.