CLRg property in metric spaces

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***ABSTRACT*: In this paper we prove a common fixed point theorem for a pair of weakly compatible maps in a metric space using CLRg property. Our proved result extend and generalize multitude of common fixed point theorems existing in the literature.**

***Keywords*: Weak compatible mappings, property (E.A), CLRg property.**

1. INTRODUCTION

Aamri *et al.* [1] generalized the concepts of non compatibility by defining property (*E.A.*). Sintunavarat *et al*. [5] introduced the notion of *CLRg* property. The concept of *CLRg* does not require a more natural condition of closeness of range.

The aim of this paper is to prove a common fixed point theorem for a pair of weakly compatible maps in a metric space using CLRg property. Our proved result extend and generalize multitude of common fixed point theorems existing in the literature. Some of results in metric spaces may be seen in [2, 4, 6] and [7].

2. PRELIMINARIES

In this section we give some preliminary ideas and definitions which are needed for our discussion.

Definition 2.1.[5] Let X be a nonempty set such that the map d: $X\times X \to$ R satisfies the following conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, z) + d(z, y) \geq d(x, y)$ for all $x, y, z \in X$.

Then d is called a metric on X, and (X, d) is called a metric space.

Definition 2.2.[5] Let (X, d) metric space and $x \in X$. Then sequence $\{x_n\}$ sequence is

(i) convergent if for every c > 0, there is a natural number N such that $d(x_{\text{n}}, x) < c$, for all $n > N$. We write it as $\lim_{n\to\infty} x_n = x$.

(ii) a Cauchy sequence, if for every $0 \prec c$, there is a natural number N such that $d(x_{\text{n}}, x_{\text{m}}) < c$, for all $n, m > N$.

Definition 2.3.[3, 7] A pair of self-maps f and g of a metric space are weakly compatible if $fgx = gfx$ for all $x \in X$ at which $fx = gx$.

Definition 2.4.[7] A pair of self maps f and g on a metric space (X, d) satisfies the property (*E.A.*) if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

The class of maps satisfying property (*E.A*) contains the class of compatible

Definition 2.5.[5] A pair of self maps f and g on a metric space (X, d) satisfies the common limit in the range of g property (*CLRg*) if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz$ for some $z \in X$.

3. MAIN RESULTS

Let us now go through our main theorem.

Theorem 3.1. Let (X, d) be a metric space. Suppose that the mappings $f, g : X\to X$ be weakly compatible self- mappings of X satisfying the contractive condition

(3.1)

$$d(fx, fy) \leq k\,[d(fx, gy) + d(fy, gx) + d(fx, gx) + d(fy, gy)]$$

for all $x, y \in X$ where $k \in [0,1/4)$ is a constant. If f and g satisfy *CLRg* property then f and g have a unique common fixed point.

Proof. Since f and g satisfy the *CLRg* property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

First we claim that $gx = fx$. Suppose not, then from (3.1),

$d(fx_n, fx) \leq k\,[d(fx_n,gx) + d(fx,gx_n) + d(fx_n,gx_n) + d(fx,gx)]$.

By making $n\to\infty$, we have

$d(gx, fx) \leq k\,[d(gx,gx) + d(fx,gx) + d(gx,gx) + d(fx,gx)]$

$= 2kd(fx,gx) < d(fx,gx)$

a contradiction, hence, $gx = fx$.

Now, let $w = fx = gx$. Since f and g are weakly compatible mappings, $fgx = gfx$ which implies that $fw = fgx = gfx = gw$.

Now, we claim that $fw = w$. Suppose not, then by (3.1), we have

$$\begin{aligned} d(fw,w) &= d(fw,fx) \\ &\leq k\,[d(fw,gx) + d(fx,gw) + d(fw,gw) + d(fx,gx)] \\ &= k\,[d(fw,gx) + d(fx,gw)] \\ &= k\,[d(gw,fx) + d(fx,gw)] \\ &= k\,[d(fw,w) + d(w,fw)] = 2k\, d(fw,w) < d(fw,w) \end{aligned}$$

a contradiction, hence, $fw = w = gw$.

Hence w is a common fixed point of f and g.

For uniqueness, we suppose that z is another common fixed point of f and g in X. Then, we have

$$\begin{aligned} d(z, w) &= d(fz, fw) \\ &\leq k\,[d(fz, gw) + d(fw, gz) + d(fw, gw) + d(fz, gz)] \\ &= k\,[d(z, w) + d(w, z) + d(w, w) + d(z, z)] \\ &= 2k\, d(z, w) < d(z, w) \end{aligned}$$

a contradiction, hence, $z = w$. Therefore, f and g have a unique common fixed point.

Theorem 3.2. Let (X, d) be a metric space and let $f, g : X \rightarrow X$ be mappings such that

(3.2)

$$\begin{aligned} d(fx, fy) \leq\ & a_1\, d(fx,gx) + a_2\, d(fy,gy) + a_3\, d(fy,gx) + \\ & a_4\, d(fx,gy) + a_5\, d(gy,gx) \end{aligned}$$

for all $x, y \in X$ where $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Suppose f and g are weakly compatible and satisfy $CLRg$ property then the mappings f and g have a unique common fixed point.

Proof. Since f and g satisfy the $CLRg$ property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\rightarrow\infty} fx_n = \lim_{n\rightarrow\infty} gx_n = gx$ for some $x \in X$.

First we claim that $gx = fx$. Suppose not, then from (3.2) we have,

$$\begin{aligned} d(fx_n, fx) \leq\ & a_1\, d(fx_n, gx_n) + a_2\, d(fx, gx) + a_3\, d(fx, gx_n) + \\ & a_4\, d(fx_n, gx) + a_5\, d(gx, gx_n). \end{aligned}$$

Making limit as $n\rightarrow\infty$, we have

$$\begin{aligned} d(gx, fx) \leq\ & a_1\, d(gx, gx) + a_2\, d(fx, gx) + a_3\, d(fx, gx) + \\ & a_4\, d(gx, gx) + a_5\, d(gx, gx) \\ =\ & (a_2 + a_3)\, d(fx,gx), \end{aligned}$$

which implies that, $[1- (a_2+ a_3)]\, d(fx,gx) \leq 0$,

which gives us, $d(fx, gx) \leq 0$, a contradiction. Hence, $gx = fx$.

Now let $z = fx = gx$. Since f and g are weakly compatible mappings $fgx = gfx$ which implies that $fz = fgx = gfx = gz$.

We claim that $gz = z$. Suppose not, then by (3.2), we have

Here
$$\begin{aligned} d(gz,z) &= d(fz,fx) \\ &\leq a_1\, d(fz, gz) + a_2\, d(fx,gx) + a_3\, d(fx,gz) + \\ &\qquad a_4\, d(fz,gx) + a_5\, d(gx, gz) \\ &= (a_3 + a_4 + a_5)\, d(fz, fx) \\ &= (a_3 + a_4 + a_5)\, d(gz, z), \end{aligned}$$

which implies that, $[\,1- (a_3+a_4 +a_5)\,]\, d(gz, z) \leq 0$

which gives us, $d(gz, z) \leq 0$, a contradiction, hence, $gz = z = fz$.

Hence z is a common fixed point of f and g.

For uniqueness, let w is another common fixed point of f and g in X. Then, we have

$$d(w, z) = d(fw, fz)$$

$$\leq a_1\, d(fw, gw) + a_2\, d(fz, gz) + a_3\, d(fz, gw) + a_4\, d(fw, gz) + a_5\, d(gz, gw)$$

$$= (a_3 + a_4 + a_5)\, d(w, z),$$

which implies that, $[\, 1 - (a_3 + a_4 + a_5)\,]\, d(w, z) \leq 0$,

that is, $d(w, z) \leq 0$,

a contradiction, hence, $z = w$. Therefore, f and g have a unique common fixed point.

Example 3.1. Let $X = [0, 1]$ and $d(x, y) = |x - y| + i|x - y|$ and the mappings $f, g : X \rightarrow X$ be defined by

$$fx = (1 + x) / 5 \text{ and } gx = x \text{ for all } x \in X .$$

Then f and g satisfies all the condition of the Theorem 3.1 by taking $k = 1/8$ and $x = 1/4$ is the common fixed point theorem of f and g.

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