Product cordiality of Path union of shell related graph

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***Abstract*: In this paper we discuss path union obtained from shell graph and shell graph with fused pendent edges to it. We show that $P_m(G')$ where $G' = S_4$, Bull (S_4), S_4^+,S_4 with two pendent vertices at a point etc and show that they are product cordial graphs under respective conditions.**

***Keywords*: labeling, cordial, product, bull graph, crown, tail graph.**

Subject Classification: 05C78

Introduction: The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Graph Theory by Harary [9], A dynamic survey of graph labeling by J.Gallian [8] and Clark, Holton.[6]. I.Cahit introduced the concept of cordial labeling [7].There are variety of cordial labeling available in labeling of graphs. Sundaram, Ponraj, and Somasundaram [10] introduced the notion of product cordial labeling. A product cordial labeling of a graph G with vertex set V is a function f from V to {0, 1} such that if each edge (uv) is assigned the label f(u)f(v), the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph. We use $v_f(0,1) = (a, b)$ to denote the number of vertices with label 1 are a in number and the number of vertices with label 0 are b in number. Similar notion on edges follows for $e_f(0,1) = (x, y)$.
A lot of work is done in this type of labeling so far. One interested in survey may refer Dynamic survey in Graph labeling by J. Gallian [8].We mention a very short part of it. Sundaram, Ponraj, and Somasundaram have shown that trees; unicyclic graphs of odd order; triangular snakes; dragons; helms; Pm∪Pn; Cm∪Pn; Pm∪K1,n; Wm∪Fn (Fn is the fan Pn+K1); K1,m∪K1,n; Wm∪ K1,n; Wm ∪Pn; Wm ∪Cn; the total graph of Pn (the total graph of Pn has vertex set V (Pn)∪E(Pn) with two vertices adjacent whenever they are neighbors in Pn); C_n if and only if n is odd; $C_n^{(t)}$, the one-point union of t copies of C_n, provided t is even or both t and n are even; K2+mK1 if and only if m is odd; $C_m \cup P_n$ if and only if m+n is odd; $K_{m,n} \cup P_s$ if s > mn; Cn+2∪K1,n; Kn∪Kn,(n−1)/2 when n is odd; Kn∪Kn−1,n/2 when n is even; and P2 n if and only if n is odd. They also prove that $K_{m,n}$ (m,n > 2), $P_m \times P_n$ (m,n > 2) and wheels are not product cordial and if a (p,q)-graph is product cordial graph, then q 6 (p−1)(p + 1)/4 + 1.
In this paper we show that path union $P_m(G')$ where $G' = S_4$, Bull (S_4), S_4^+, S_n^{++},S_4 with two pendent vertices at a point etc are product cordial graphs and obtain the condition for same.

Preliminaries:
3.1 Fusion of vertex. Let G be a (p,q) graph. Let u≠v be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted. The new graph has p-1vertices and at least q-1 edges. If $u \epsilon G_1$ and $v \epsilon G_2$, where G_1 is (p_1,q_1) and G_2 is (p_2,q_2) graph. Take a new vertex w and all the edges incident to u and v are joined to w and vertices u and v are deleted. The new graph has p_1+p_2-1 vertices and $q_1 + q_2$ edges. Sometimes this is referred as u is identified with the concept is well elaborated in John Clark, Holton[6]
3.2 Crown graph. It is C_n □K_2.At each vertex of cycle a n edge was attached. We develop the concept further to obtain crown for any graph. Thus crown (G) is a graph G □K_2.It has a pendent edge attached to each of it's vertex. If G is a (p,q) graph then crown(G) has q+p edges and 2p vertices.
3.3 Flag of a graph G denoted by FL(G) is obtained by taking a graph G=G(p, q).At suitable vertex of G attach a pendent edge. It has p+1 vertices and q+1 edges.
3.4 A bull graph bull(G) was initially defined for a C_3-bull.It has a copy of G with an pendent edge each fused with any two adjacent vertices of G. For G is a (p,q) graph, bull(G) has p+2 vertices and q+2 edges.
3.5 A tail graph (also called as antenna graph) is obtained by fusing a path p_k to some vertex of G. This is denoted by tail(G, P_k). If there are t number of tails of equal length say (k-1) then it is denoted by tail(G, tp_k). If G is a (p,q) graph and a tail P_k is attached to it then tail(G, P_k) has p+k-1 vertices and q+k-1 edges4.
3.6 Path union of G ,i.e.(G) is obtained by taking a path p_m and take m copies of graph G . Then fuse a copy each of G at every vertex of path at given fixed point on G. It has mp vertices and mq +m-1 edges. Where G is a (p, q) graph.

2. Main Results:
Theorem 4.1 $P_m(S_n)$ is product cordial iff m is even number.
Proof: The path P_m is defined as (v_1, e_1, v_2, e_2, ..,v_m). The copy of S_n fused at i^{th} vertex of P_m is defined as : the cycle C_4 of S_4 as ($u_{i,1}$,$c_{i,1}$,$u_{i,2}$,$c_{i,2}$,$u_{i,3}$,$c_{i,3}$,$u_{i,4}$,$c_{i,4}$,$u_{i,1}$); the chord ($u_{i,1}u_{i,3}$) ; i = 1, 2, ..m. Note that $u_{i,1}$ is v_i, i = 1, 2, ..m.
Define f:V(G)→{0,1} as follows.
Case i = 2x.
f ($u_{i,j}$) = 0 for all i = 1, 2, ..x and j = 1, 2, 3, 4.;
f($u_{i,j}$) = 1 for i= x+1,x+2,..2x, and j = 1, 2, 3, 4.The label number distribution is $v_f(0,1)$= (4x,4x) ; $e_f(0,1)$ =(6x,6x-1).If we change the vertex on S_4 to three degree vertex on S_4 , we get product cordial path union with the same f .

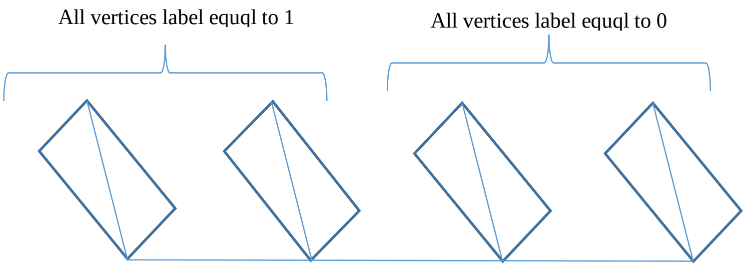

Fig 4.1: $P_4(S_4)$: product cordial graph : $v_f(0,1)= (8,8)$; $e_f(0,1) =(12,11)$.

Case m= 2x+1.

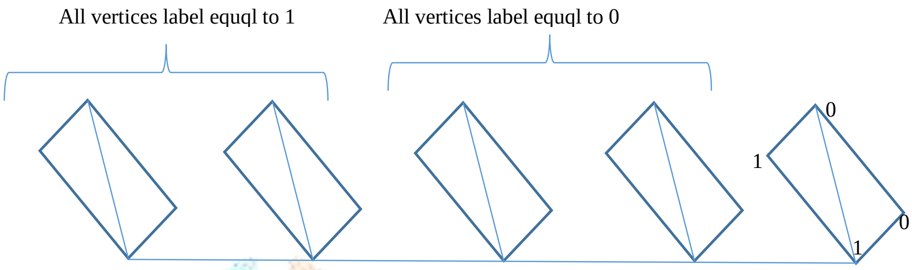

Fig 4.2: $P_4(S_4)$: not product cordial graph : $v_f(0,1)= (10,10)$; $e_f(0,1) =(16,13)$.

If we try to fulfill the condition $|v_f(0) - v_f(1)| \leq 1$ on vertices the condition for edges is spoiled. Even if we change the point of contact of P_m and S_4 from 3-degree vertex to 2-degree vertex, there is no f :V(G)→{0,1} that will label $P_{(2x+1)}(s_4)$ as product cordial.

Thus the graph $P_{2x+1}(S_4)$ is not product cordial. #

Theorem 4.2. Let G' be a flag graph $FL(S_4)$,then path union of G' given by $P_m(G')$ is product cordial for all m .

Proof: The path P_m is defined as $(v_1, e_1, v_2, e_2, .., v_m)$. The copy of S_4 fused at i^{th} vertex of P_m is defined as : the cycle C_4 of S_4 as $(u_{i,1},c_{i,1},u_{i,2},c_{i,2},u_{i,3}, c_{i,3}, u_{i,4},c_{i,4},u_{i,1})U\{u_{i,5}\}$, the chord $(u_{i,1}u_{i,3})$; ; i = 1, 2, ..m. Note that $u_{i,1}$ is v_i, i = 1, 2, ..m. Further in structure 1 the pendent vertex is attached at 2-degree vertex of S_4 by edge $(u_{i,2}u_{i,5})$ or by edge $(u_{i,4}u_{i,5})$ .when the pendent vertex is attached at degree 3 vertex of S_4 by edge $(u_{i,3}u_{i,5})$ or by $(u_{i,1}u_{i,5})$, we call it as structure 2. Further v_i is same as $u_{i,1}$.
Define f: V(G)→{0,1} as follows,

Case i = 2x.

$f(u_{i,j})$ =0 for all i = 1, 2, ..x and j = 1, 2, 3, 4, 5;

$f(u_{i,j})$ =1 for i= x+1,x+2,..2x, and j = 1, 2, 3, 4, 5.

The label number distribution is $v_f(0,1)= (5x,5x)$; $e_f(0,1) =(7x,7x-1)$.

Case i = 2x+1

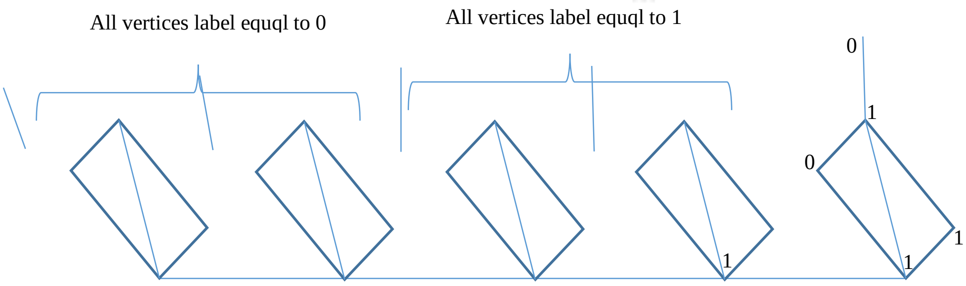

Fig 4.3: $P_5(FL(S_4))$: product cordial graph : $v_f(0,1)= (12,12)$; $e_f(0,1) =(14,13)$.

To obtain a labeled copy of $P_{2x+1}(FL(S_4)$ we first follow the labeling on $P_{2x}(FL(S_4)$ part as given above.

For i = 2x+1 , $f(u_{i,j})$ = 0; j = 2,5.

$f(u_{i,j})$ =1 for j = 1, 3, 4,

The label number distribution is $v_f(0,1)= (5x+2,5x+3)$; $e_f(0,1) =(7x+3,7x+3)$.

Theorem 4.3 Let G' be a bull graph bull(S_4),then path union of G' given by P_m(G') is product cordial for all m.

Proof: The path P_m is defined as (v_1, e_1, v_2, e_2, ..,v_m). The copy of S_4 fused at i^{th} vertex of P_m is defined as : the cycle C_4 of S_4 as ($u_{i,1}$,$c_{i,1}$,$u_{i,2}$,$c_{i,2}$,$u_{i,3}$, $c_{i,3,}$ $u_{i,4}$,$c_{i,4}$,$u_{i,1}$)U{$u_{i,5}$,$u_{i,6}$ } ; i = 1, 2, ..m. Note that $u_{i,1}$ is v_i, i = 1, 2, ..m. Further in structure 1 the pendent vertices are attached at $u_{i,2}$ and $u_{i,3}$ of S_4 by edges ($u_{i,2}u_{i,5}$) and by edge ($u_{i,3}u_{i,6}$) .when the pendent vertex is attached at degree 3 vertex of S_4 by edge ($u_{i,1}u_{i,5}$) or by ($u_{i,3}u_{i,6}$), we call it as structure 2. Further v_i is same as $u_{i,1}$.

Define f: V(G)→{0,1} as follows, |V(G)|= 6m,|E(G)|= 8m-1.

Case i = 2x.

f($u_{i,j}$) =0 for all i = 1, 2, ..x and j = 1, 2, 3, 4, 5,6

f($u_{i,j}$) =1 for i= x+1,x+2,..2x, and j = 1, 2, 3, 4, 5,6

The label number distribution is v_f(0,1)= (6x,6x) ; e_f(0,1) =(8x,8x-1).

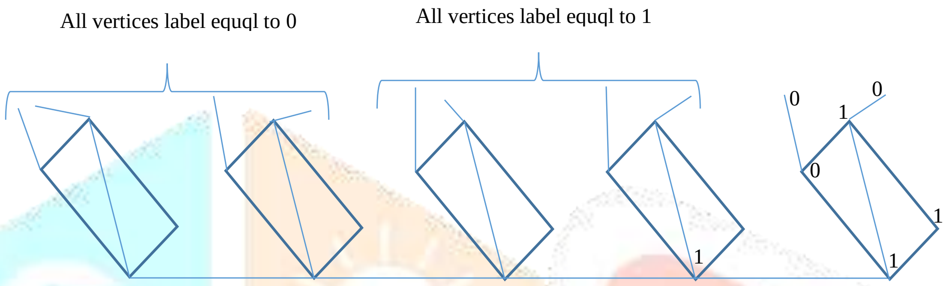

Fig 4.4: P_5(bull(S_4): product cordial graph :
v_f(0,1)= (15,15) ; e_f(0,1) =(20,19).

Case m = 2x+1

To obtain a labeled copy of P_{2x+1}(bull(S_4) we first follow the labeling on P_{2x}(bull(S_4) part as given above. For i= 2x+1 we have,

f($u_{i,j}$) =1 for j = 1, 3, 4,

f($u_{i,j}$) =0 for j = 2, 5, 6 .

The label number distribution is v_f(0,1)= (6x+3,6x+3) ; e_f(0,1) =(8x+4,8x+3).

Theorem 4.4 Let G' crown on S_4, given by S_4^+ then path union of G' given by P_m(G') is product cordial for all m.

Proof: The path P_m is defined as (v_1, e_1, v_2, e_2, .., v_m). The copy of S_4 fused at i^{th} vertex of P_m is defined as : the cycle C_4 of S_4 as ($u_{i,1}$,$c_{i,1}$,$u_{i,2}$,$c_{i,2}$,$u_{i,3}$, $c_{i,3,}$ $u_{i,4}$,$c_{i,4}$,$u_{i,1}$)U{$u_{i,5}$,$u_{i,6}$, $u_{i,7}$, $u_{i,8}$ }, the chord ($u_{i,1}u_{i,3}$), i = 1, 2, ..m. Note that $u_{i,1}$ is v_i; i = 1, 2, ..m. the pendent edges are attached at $u_{i,1}$, $u_{i,2}$ and $u_{i,3}$, $u_{i,4}$ and are given by ($u_{i,1}u_{i,5}$) , ($u_{i,2}u_{i,6}$) , $u_{i,3}u_{i,7}$) and edge ($u_{i,4}u_{i,8}$) . Further v_i is same as $u_{i,1}$.

Define f:V(G)→{0,1} as follows,

Case i = 2x.

f($u_{i,j}$) =1 for all i = 1, 2, ..x and j = 1, 2, 3, 4, 5, 6,7,8.

f($u_{i,j}$) =0 for i= x+1,x+2,..2x, and j = 1, 2, 3, 4, 5, 6, 7, 8.

The label number distribution is v_f(0,1)= (8x,8x) ; e_f(0,1) =(10x,10x-1).

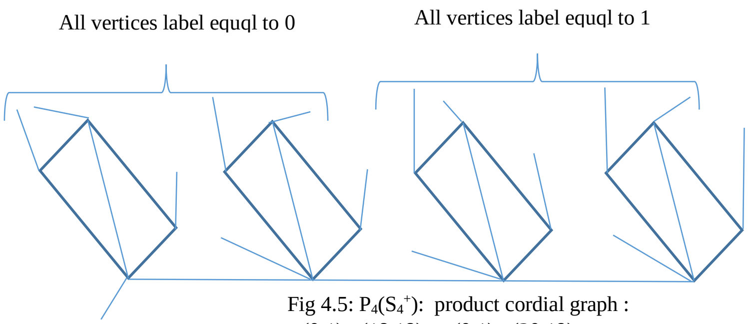

Fig 4.5: $P_4(S_4^+)$: product cordial graph :
v_f(0,1)= (18,18) ; e_f(0,1) =(20,19).

Case m = 2x+1. To obtain a labeled copy of $P_{2x+1}(S_4^+)$ we first follow the labeling on $P_{2x}(S_4^+)$ part as given above. i = 2x+1

$f(u_{i,j})$ =1 for j = 1, 2, 3, 4,

$f(u_{i,j})$ =0 for j = 5, 6 , 7, 8

The label number distribution is $v_f(0,1)$= (8x+4,8x+4) ; $e_f(0,1)$ =(10x+5,10x+4).

Thus the graph G is product cordial for all m #.

Theorem 4.5 Let G' be tail(S_4,2P_2) then path union of G' given by G = P_m(G') is product cordial for all m. Proof: The path P_m is defined as (v_1, e_1, v_2, e_2, ..,v_m). The copy of S_4 fused at i[th] vertex of P_m is defined as : the cycle C_4 of S_4 as ($u_{i,1}$,$c_{i,1}$,$u_{i,2}$,$c_{i,2}$,$u_{i,3}$, $c_{i,3}$, $u_{i,4}$,$c_{i,4}$,$u_{i,1}$)U{$u_{i,5}$,$u_{i,6}$,} , the chord ($u_{i,1}u_{i,3}$), i = 1, 2, ..m. Note that $u_{i,1}$ is v_i; i = 1, 2, ..m; the pendent edges are attached at $u_{i,1}$ are ($u_{i,1}u_{i,5}$) , ($u_{i,1}u_{i,6}$).. Note that |V(G)|= 6m; |E(G)|= 8m-1

Define f:V(G)→{0,1} as follows,

Case m = 2x.

$f(u_{i,j})$ = 0 for all i = 1, 2, ..x and j = 1, 2, 3, 4, 5,6

$f(u_{i,j})$ = 1 for i= x+1,x+2,..2x, and j = 1, 2, 3, 4, 5,6

The label number distribution is $v_f(0,1)$= (6x,6x) ; $e_f(0,1)$ =(8x,8x-1).

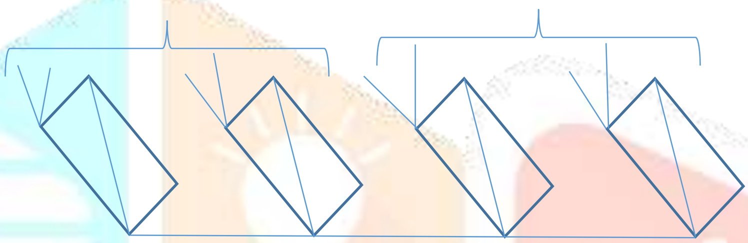

Fig 4.6: P_4(tail(S_4,2P_2)): product cordial graph : $v_f(0,1)$= (12,12) ; $e_f(0,1)$ =(16,15).

Case m = 2x+1

To obtain a labeled copy of P_{2x+1}(tail(S_4,2P_2)) we first follow the labeling on P_{2x}(tail(S_4.2P_2)) part as given above.

For i = 2x+1 $f(u_{i,j})$ = 1 j = 1, 3, 4,

$f(u_{i,j})$ = 0 j = 2, 5, 6

The label number distribution is $v_f(0,1)$= (6x+3,6x+3) ; $e_f(0,1)$ =(8x+4,8x+3).#

Theorem 4.6 Let G' be a graph obtained from S_4 by fusing 2 pendent edges each at one pair of adjacent vertices. Then path union of G' given by G = P_m(G') is product cordial for all m.

Proof: The path P_m is defined as (v_1, e_1, v_2, e_2, ..,v_m). The copy of S_4 fused at i[th] vertex of P_m is defined as : the cycle C_4 of S_4 as ($u_{i,1}$,$c_{i,1}$,$u_{i,2}$,$c_{i,2}$,$u_{i,3}$, $c_{i,3}$, $u_{i,4}$,$c_{i,4}$,$u_{i,1}$)U{$u_{i,5}$, $u_{i,6}$, $u_{i,7}$, $u_{i,8}$ } , the chord ($u_{i,1}u_{i,3}$), i = 1, 2, ..m. Note that $u_{i,1}$ is v_i; i = 1, 2, ..m; the pendent edges are attached at $u_{i,1}$ are ($u_{i,1}u_{i,5}$) , ($u_{i,1}u_{i,6}$) and at $u_{i,2}$ are ($u_{i,2}u_{i,7}$) , ($u_{i,2}u_{i,8}$). Further v_i is same as $u_{i,1}$.

Note that |V(G)|= 16x for m = 2x. |E(G)|= 20x-1

Define f:V(G)→{0,1} as follows,

Case i = 2x.

$f(u_{i,j})$ = 0 for all i = 1, 2, ..x and j = 1, 2, 3, 4, 5,6,7,8

$f(u_{i,j})$ = 1 for i= x+1,x+2,..2x, and j = 1, 2, 3, 4, 5,6 ,7,8

The label number distribution is $v_f(0,1)$= (8x,8x) ; $e_f(0,1)$ =(10x,10x-1).

Case m = 2x+1

To obtain a labeled copy of P_{2x+1}(G') we first follow the labeling on P_{2x}(G') part as given above.

For i = 2x+1

$f(u_{i,j}) = 1$ for j = 1, 3, 4, 5

$f(u_{i,j}) = 0$ for j = 2,6 ,7,8

The label number distribution is $v_f(0,1)= (8x+4,8x+4)$; $e_f(0,1) =(10x+5,10x+4)$.

Theorem 4.7 Let G' be a graph obtained from S_4 by fusing 2 pendent edges each at every vertex of S_4. Then path union of G' given by G = P_m(G') is product cordial for all m.

Proof: The path P_m is defined as (v_1, e_1, v_2, e_2, ..,v_m). The copy of S_4 fused at i^{th} vertex of P_m is defined as : the cycle C_4 of S_4 as $(u_{i,1},c_{i,1},u_{i,2},c_{i,2},u_{i,3}, c_{i,3}, u_{i,4},c_{i,4} ,u_{i,1})U\{u_{i,5}, u_{i,6} , u_{i,7}, u_{i,8} , u_{i,9}, u_{i,10} , u_{i,11}, u_{i,12}\}$, the chord $(u_{i,1}u_{i,3})$, i = 1, 2, ..m. Note that $u_{i,1}$ is v_i; i = 1, 2, ..m; the pendent edges are attached at $u_{i,1}$ are $(u_{i,1}u_{i,5})$, $(u_{i,1}u_{i,6})$ and at $u_{i,2}$ are $(u_{i,2}u_{i,7})$, $(u_{i,2}u_{i,8})$.

Note that |V(G)|= 12m and |E(G)|= 14m-1

Define f:V(G)→{0,1} as follows,

Case i = 2x.

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x and j = 1, 2, ,..12.

$f(u_{i,j}) = 0$ for i= x+1,x+2,..2x, and j = 1, 2, ,…12.

The label number distribution is $v_f(0,1)= (12x,12x)$; $e_f(0,1) =(14x,14x-1)$.

Case i = 2x+1

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x and j = 1, 2, …12,

$f(u_{i,j}) = 0$ for i= x+1,x+2,..2x, and j = 1, 2….12;

$(u_{i,j}) = 1$ for i = x+1 and j = 1, 2, 3, 4,5,6;

$f(u_{i,j}) = 0$ for i= x+1; j =7, 8, …12.

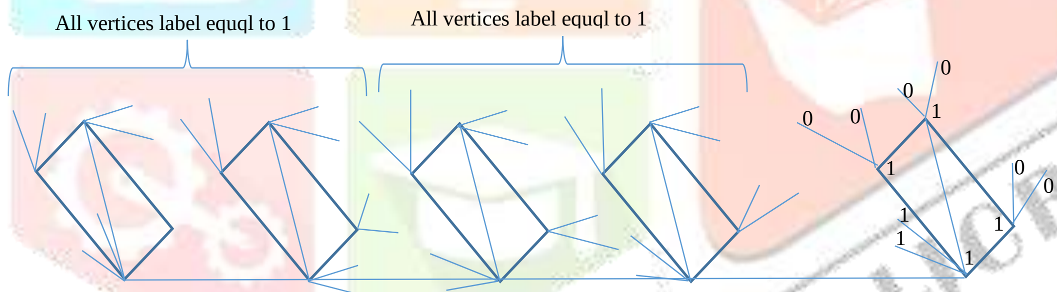

Fig 4.6: P_4(tail(S_4,2P_2)): product cordial graph : $v_f(0,1)= (30,30)$; $e_f(0,1) =(35,34)$.

The label number distribution is $v_f(0,1)= (12x+6,12x+6)$; $e_f(0,1) =(14x+7,14x+6)$.

Conclusions: In this paper we discuss path union graph P_m(G) where G is obtained from S4 by attaching up to two pendent vertices at each vertex of S_4.Weshow that :

1) $P_m(S_n)$ is product cordial iff m is even number.

2) Let G' be a flag graph FL(S_4),then path union of G' given by P_m(G') is product cordial for all m.

3) Let G' be a bull graph bull(S_4),then path union of G' given by P_m(G') is product cordial for for all

4) Let G' crown on S_4, given by S_4^+ then path union of G' given by P_m(G') is product cordial for all m.

5) Let G' be tail(S_4,2P_2) then path union of G' given by G = P_m(G') is product cordial for for all m.

6) Let G' be a graph obtained from S_4 by fusing 2 pendent edges each at one pair of adjacent vertices. Then path union of G' given by G = P_m(G') is product cordial for even m only.

7) Let G' be a graph obtained from S_4 by fusing 2 pendent edges each at every vertex of S_4. Then path union of G' given by G = P_m(G') is product cordial for all m.

These results shows that path unions taken on S_4^{+t} are product cordial for all m (t =1,2) and all other path unions taken on G such that G is not isomorphic to S_4^{+t} for some t are product cordial for even m only. This tempts us to say that $P_m(S_4^{+t})$ for all t and all m are product cordial.

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