COMPUTATIONAL SCHEME BY GALERKIN METHOD FOR TWO POINT BOUNDARY VALUE PROBLEMS BY QUINITIC B-SPLINES

[1]K. Sharath babu, [2]B. Ravindra Reddy, [3]Venkata Brahman, [4]G. Gangadhar, [5]N. Amaranth
[1]Matrusri Engineering College, Hyderabad, Telangana, India, [2]JNTUH College of Engineering, Hyderabad, Telangana, India, [3]Krishna Univeristy , Andhra Pradesh, India, [4,5]JNTU, Hyderabad, Telangana, India.

Abstract: The fourth order boundary value problems arise in the mathematical modeling such as visco-elastic and inelastic flow problems, bending of beams and plates deflection theory , beam element theory and many applications of engineering and applied sciences. Computing such type of boundary value problems analytically is possible only in rare cases. Many researchers worked for the computational solutions of fourth order boundary value problems. In the present work, we attempted to present a simple finite element method which involves Galerkin approach with quintic B-splines as basis functions to solve a fourth order boundary value problem with suitable boundary conditions.

Key Words: **Two point boundary value problem, Basis functions, Quintic B-Splines, Galerkian method.**

I. INTRODUCTION

Many a times the fourth order boundary value problems arise in the mathematical modeling of visco-elastic and inelastic flow problems, bending of beams and plates deflection theory , beam element theory and many applications of engineering and applied mathematics. Computing such type of boundary value problems analytically is possible only in rare cases. Many researchers worked for the computational solutions of fourth order boundary value problems. In the present research work, we are attempted to present a simple finite element method which involves Galerkin approach with quintic B-splines as basis functions to solve a fourth order boundary value problem with suitable boundary conditions. In this paper we consider a general fourth order linear boundary value problem with mixed boundary conditions given by

$$a_0(x)y^{(4)}(x)+a_1(x)y'''(x)+a_2(x)y''(x)+a_3(x)y'(x)+a_4(x)y(x)=b(x),\ a<x<b \tag{1}$$

Subject to the mixed boundary conditions

$$y(a)= A_0,\ y(b) = B_0,\ y^{I}(a)=A_1\ ,\ y^{I}(b) = B_1 \tag{2}$$

Where A_0, A_1, B_0 and B_1 and are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, and $b(x)$, are all smooth functions defined in the interval [a, b]. The boundary value problem (1) is solved with the boundary conditions (2). To solve the boundary value problem (1) by the Galerkin method with quintic B-splines, as basis functions, we approximate $y(x)$ as

$$y(x) =\sum_{j=-2}^{n+2} \lambda_j\ B_j(x) \tag{3}$$

Where λ_j's are the nodal parameters to be determined.

$$A_0 = y(x_0)=\lambda_{-2}\ B_{-2}(x_0)+ \lambda_{-1}\ B_{-1}(x_0) +\lambda_0 B_0(x_0) +\lambda_1\ B_1(x_0)+ \lambda_2\ B_2(x_0) \tag{4}$$

$$B_0 = y(x_n)= \lambda_{n-2}\ B_{n-2}(x_n)+ \lambda_{n-1}\ B_{n-1}(x_n) +\lambda_n B_n(x_n) +\lambda\ _{Bn+1}(x_n)+ \lambda_2\ B_{n+2}\ (x_n) \tag{5}$$

Eliminating λ_{-2} and λ_{n-2} from the equations (3),(4) and (5) , we get

$$Y(x) = w(x)+\sum_{j=-1}^{n+1} \lambda_j\ \tilde{B}_j(x) \tag{6}$$

Where $w(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x_0)+ \frac{B_0}{B_{n+2}(x_n)} B_{n+2}(x_0)$ (7)

and

$$\tilde{B}_j(x) = \begin{cases} B_j(x) - \frac{B_j(x_0)}{B_{-2}(x_0)} B_{-2}(x), & j = -1,0,1,2 \\ B_j(x), & j = 3,4,\dots\dots,n-3 \\ B_j(x) - \frac{B_j(x_n)}{B_{n+2}(x_0)} B_{n+2}(x), & j = n-2,n-1,n,n+1 \end{cases} \tag{8}$$

Applying Galerkin method for (1) with the basis functions $\tilde{B}_i(x)$, i=-1,0,1,….n,n+1, we get

$$\int_{x_0}^{x_n}[a_0(x)y^{(4)} + a_1(x)\ y''' +a_2(x)\ y'' + a_3(x)\ y' +a_4(x)\ y]\ \tilde{B}_i(x)\ dx = \int_{x_0}^{x_n} b(x)\ \tilde{B}_i(x)dx$$

for i= -1,0,1…… ,n,n+1. (9)

In Galerkin method the basis functions should vanish on the boundary where the mixed boundary conditions are specified. In the set of quintic B-splines .$\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), . . . B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$, the basis functions $B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), B_2(x), B_{n-2}(x), B_{n-1}(x), B_n(x)$, are not eliminating at one of the boundary points. So, there is a necessity of redefining the Basis functions into a new set of basis functions which vanish on the boundary where the Boundary conditions are specified. For this, we proceed in the following manner. Using the quintic B-splines[1,6] and the mixed type of boundary conditions prescribed in (2), we get the approximate solution at the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the given type of boundary conditions is specified. For this, we used the quintic B-splines and the mixed type of boundary conditions prescribed in (2), we get the approximate solution by solving the matrix form of the equation

$$AX = B \tag{10}$$

II. METHODOLOGY

Divide the space variable domain [a, b] of the system (1) and (2) into 'n' subintervals by means of n+1 distinct points $x_0, x_1, x_2, \ldots, x_n$ such that $a = x_0 < x_1 < x_2 < . . . < x_{n-1} < x_n = b$.Introduce ten additional knots $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ and x_{n+5} with equal interval width .

An Integral element in AX=B is

$$\sum_{m=0}^{n-1} I_m \quad \text{Where } I_m = \int_{x_m}^{x_{m+1}} r_i(x) r_j(x) Z(x) dx$$

here $r_i(x)$ and $r_j(x)$ with width are the basis functions or their derivatives. It may be noted that $I_m = 0$ if $(x_{i-3}, x_{i+3}) \cap (x_{j-3}, x_{j+3}) \cap (x_m, x_{m+1}) = \emptyset$.To evaluate each I_m, Here employed the 6-point Gauss-Legendre quadrature technique. Thus the stiff matrix A is an eleven diagonal band matrix which must be non-singular. The nodal parameter vector λ has been obtained from the system

$A\lambda = b$ using a band matrix solution methodology.

2.1 Numerical illustrations

To test the applicability of the proposed method, we considered two linear boundary value problems because the accurate solutions for these problems are available in the literature. For the examples, the solutions obtained by the proposed method are compared with the exact solutions available in the literature.

Example 1.
Consider the linear boundary value problem

$$y^{(4)} + 4y = 1, \quad -1 < x < 1 \tag{11}$$

Applying the conditions y(-1)=y(1)=0,

$$y^1(-1) = -y^1(1) = 0.25\left(\frac{sinh2 - sin2}{cosh2 + cos2}\right)$$

The exact solution is

$$y = 0.25\{\,1 - \frac{2sinh1sin1sinhxsinx + coh1cos1coshxcosx}{cosh2 + cos2}\}$$

The proposed method mentioned is tested on this problem where the domain [−1, 1] is divided into 10 equal sub-intervals. Numerical results for this problem are given in Table 1.1. The maximum absolute error obtained by the proposed method is 1.3034×10^{-6}.

Example 2.

Consider the linear boundary value problem

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x, \; 0<x<1 \tag{12}$$

With the prescribed mixed boundary conditions $y(0)=y(1)=y^1(0) =1$,$y^1(1) = - 2.718$

The exact solution of equation (12) is $y =(x-x^2)\, e^x$, $0<x<1$. This problem numerically solved each mesh point by using the proposed method. The selected method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. Numerical results for this problem are given in Table 1.2. The maximum absolute error obtained by the proposed method is 5.99×10^{-6}.

III. RESULT & DISCUSSION

In this research work , we have developed a Galerkin method with quintic B-splines as basis functions to solve a fourth order boundary value problems with boundary conditions. The quintic B-spline basis set has been redefined into a new set of basis functions which vanish on the boundary. This method is tested to solve few numbers of problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature.

Table1.1: Numerical results for Example 1.1

x	Exact solution	Absolute error by proposed method
-0.8	0.0397692	1.788139×10^{-07}
-0.6	0.0749849	5.736947×10^{-07}
-.04	0.1023106	1.288950×10^{-06}
-0.2	0.1195382	9.983778×10^{-07}
0.0	0.1254157	8.493662×10^{-07}
0.2	0.1195382	1.303852×10^{-06}
0.4	0.1023106	$9.387732 \quad 10^{-07}$
0.6	0.0749849	$9.387732 \quad 10^{-07}$
0.8	0.0397692	$6.556511 \quad 10^{-07}$

Table1.2: Numerical results for Example1.2

x	Exact solution	Absolute error by proposed method
0.1	0.0994653	$6.631017 \quad 10^{-07}$
0.2	0.1954244	$1.519918 \quad 10^{-06}$
0.3	0.2834704	$2.443790 \quad 10^{-07}$
0.4	0.3580379	$4.380941 \quad 10^{-06}$
0.5	0.4121803	$5.662441 \quad 10^{-06}$
0.6	0.4373085	$5.841255 \quad 10^{-06}$
0.7	0.4228881	$5.990267 \quad 10^{-06}$
0.8	0.3560865	$5.871058 \quad 10^{-06}$
0.9	0.2213642	$4.246831 \quad 10^{-06}$

IV. CONCLUSIONS

This is an subsidiary method to solve a fourth order differential equation computationally with the given boundary conditions at each mesh point. By using Galerkin method with the selected quinitic polynomials. The objective of this research paper is to present a simple and elegant method to solve a fourth order boundary value problem. The numerically obtained results are fairly close to the exact results.

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