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Generalized Closed Sets In Fermatean Picture Fuzzy Topological Spaces

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Abstract:

This paper introduces various generalized closed sets in Fermatean Picture Fuzzy Topological Spaces (FPFTS), including generalized closed, generalized pre-closed, generalized semi-closed and generalized α-closed sets. The fundamental properties, theorems and proofs are presented with numerical examples.

Keywords: Fermatean Picture Fuzzy generalized closed, Fermatean Picture Fuzzy generalized pre-closed, Fermatean Picture Fuzzy generalized semi-closed and Fermatean Picture Fuzzy generalized α-closed sets

1. Introduction:

The introduction of fuzzy sets by Zadeh in 1965 [19] provided a groundbreaking approach to handling the inherent vagueness and uncertainty prevalent in real-world scenarios by assigning degrees of membership to elements within a set, represented by values in the interval [0, 1]. This seminal work spurred the development of numerous extensions designed to capture more intricate forms of uncertain information.

One notable extension is Atanassov's intuitionistic fuzzy sets (IFSs) [1], which augment the concept of fuzzy sets by incorporating a degree of non-membership alongside the degree of membership, subject to the constraint that their sum does not exceed 1. This framework allows for the explicit representation of hesitation or uncertainty regarding an element's belongingness [2]. Building upon this foundation, various other generalizations have emerged to address diverse aspects of uncertainty.

The integration of fuzzy set theory with topology led to the emergence of fuzzy topology. Chang's pioneering work in 1968 [4] established the fundamental concepts of fuzzy topological spaces by defining fuzzy open sets and extending classical topological notions to this fuzzy setting. Subsequently, Sostak [11] offered an alternative definition of a fuzzy topological structure, while Coker [5] provided an introductory overview of fuzzy topological spaces. Bhattacharya [3] further contributed to this field by exploring various properties within fuzzy topology. The concepts of generalized closed sets, initially introduced in classical topology, have also been extended to the realm of fuzzy topological spaces, as evidenced by the works of Thakur and Chaturvedi [12, 13, 14] in the context of intuitionistic fuzzy topology, and Shalini and Mary [8] within Pythagorean fuzzy topological spaces (an earlier extension of fuzzy sets [18]). Kadambavanam and Vaithiyalingam [7] and Rajarajeswari and Senthil Kumar [9] have also contributed to the study of weakly generalized closed sets and generalized pre-closed sets in intuitionistic fuzzy topological spaces, respectively.

More recently, to address situations where information includes degrees of membership, non-membership, and a degree of refusal, picture fuzzy sets were introduced. Senapati and Yager [10] further extended this concept to Fermatean fuzzy sets (FFSs), offering increased flexibility in modeling uncertainty by relaxing the constraints on the sum of membership and non-membership degrees. V.Chitra and B.Maheswari have significantly contributed to the study of Fermatean picture fuzzy sets (FPFSs) [15], a further generalization, and have explored their properties related to continuity [16] and connectedness and compactness [17] in Fermatean picture fuzzy topological spaces, as introduced by Ibrahim [6]. These recent developments highlight the ongoing interest in exploring topological concepts within these richer fuzzy frameworks.

This research aims to contribute further to the understanding of topological structures within the context of investigating new types of generalized open and closed sets in Fermatean picture fuzzy topological spaces and their properties. By doing so, we seek to provide a deeper insight into the topological characteristics arising from the Fermatean picture fuzzy set framework and their potential applications in handling complex uncertain information.

2. Preliminaries

Definition 2.1. [15]

A Fermatean Picture Fuzzy (FPF) set $\mathcal{A}$ in a universe U is an object of the form,

$\mathcal{A} = \{\langle x, \langle x, \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)\rangle | x \in U\rangle\}$ where $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of x in $\mathcal{A}$ and the following conditions are satisfied

$$0 \leq \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x) \leq 1,$$

$$0 \leq {\alpha_{\mathcal{A}}}^3(x) + {\beta_{\mathcal{A}}}^3(x) + {\gamma_{\mathcal{A}}}^3(x) \leq 1, \forall x \in U.$$

Then $\forall x \in U, \pi_{\mathcal{A}}(x) = 1 - {\alpha_{\mathcal{A}}}^3(x) + {\beta_{\mathcal{A}}}^3(x) + {\gamma_{\mathcal{A}}}^3(x)$ is called the degree of refusal membership of x in $\mathcal{A}$.

When dealing with human opinions that involve multiple types of responses such as "yes," "abstain," "no," and "refusal," Fermatean Picture Fuzzy Sets offer a suitable mathematical framework to handle the complexity and uncertainty inherent in such scenarios. For example, in feedback mechanisms for products or services, users might express satisfaction (yes), dissatisfaction (no), neutrality (abstain) or refuse to provide feedback.

Definition 2.2. [15]

Let X be a non-empty set and the FPF sets A and B be in the form

$A = \{\langle x, \langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle | x \in X\rangle\}$ and $B = \{\langle x, \langle x, \alpha_B(x), \beta_B(x), \gamma_B(x)\rangle | x \in X\rangle\}$

1. $(A) \subseteq (B)$ iff $\alpha_A(x) \leq \alpha_B(x), \beta_A(x) \leq \beta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$
2. $(A) = (B)$ iff $\alpha_A(x) = \alpha_B(x), \beta_A(x) = \beta_B(x)$ and $\gamma_A(x) = \gamma_B(x) \forall x \in X$
3. $A \cap B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x)\rangle | x \in X\}$ where
   I. $\alpha_{A\cap B}(x) = min\{\alpha_A(x), \alpha_B(x)\}$
   II. $\beta_{A\cap B}(x) = min\{\beta_A(x), \beta_B(x)\}$
   III. $\gamma_{A\cap B}(x) = max\{\gamma_A(x), \gamma_B(x)\}$
4. $A \cup B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x)\rangle | x \in X\}$ where
   I. $\alpha_{A\cup B}(x) = max\{\alpha_A(x), \alpha_B(x)\}$
   II. $\beta_{A\cup B}(x) = min\{\beta_A(x), \beta_B(x)\}$
   III. $\gamma_{A\cup B}(x) = min\{\gamma_A(x), \gamma_B(x)\}$

Definition 2.3.[17]

Let X be a non-empty set and τ be a family of Fermatean Picture Fuzzy ($\mathcal{F}$) subset of X. If

1. $1_X, 0_X \in \tau$
2. for any $\mathcal{F}_1, \mathcal{F}_2 \in \tau$, we have $\mathcal{F}_1 \cap \mathcal{F}_2 \in \tau$,
3. for any $\{\mathcal{F}_i\}_{i\in I} \subset \tau$, we have $\cup_{i\in I}\mathcal{F}_i \in \tau$ where I is an arbitrary index set then τ is called a Fermatean Picture Fuzzy topology on X.

Fermatean Picture Fuzzy topological space (FPFTS) is defined as the pair (X, τ). Every element in τ is referred to as an open Fermatean Picture Fuzzy subset. A closed Fermatean Picture Fuzzy set is the complement of an open Fermatean Picture Fuzzy subset.

Definition: 2.4.[17]

$A = \{x, \alpha_{\mathcal{F}}(x), \beta_{\mathcal{F}}(x), (\gamma_{\mathcal{F}}(x)) : x \in X\}$ be a FPFS in X and (X, τ) be a FPFTS. Fermatean Picture fuzzy closure and interior are defined by

1. $FPFcl(A) = \cap\{H : H \text{ is closed Fermatean Picture fuzzy set in } X \text{ and } \mathcal{F} \subset H\}$.
2. $FPFint(A) = \cup\{G : G \text{ is open Fermatean Picture fuzzy set in } X \text{ and } G \subset \mathcal{F}\}$.

3. Generalized Closed Sets in FPFTS

Generalized Closure and Interior

Definition 3.1:

Let (X, $\mathscr{F}_p$**)** be a Fermatean Picture Fuzzy Topological Space (FPFTS)**,** and let A be a Fermatean Picture Fuzzy Set (FPFS) in $\boldsymbol{X}$. The generalized closure of $\boldsymbol{A}$, denoted as FPF-Gcl(A), is the smallest generalized closed set containing $\boldsymbol{A}$. Formally,

FPF-Gcl(A)= $\cap\{G \in \mathscr{F}_p{}^c \mid A \subseteq G\}$ where $\mathscr{F}_p{}^c$ is the collection of all generalized closed sets in X.

Definition 3.2:

The generalized interior of A, denoted as $FPF - Gint(A)$, is the largest generalized open set contained in A. Formally,

FPF-Gint(A)= $\cup\{ G \in \mathscr{F}_p{}^o \mid G \subseteq A\}$

Where $\mathscr{F}_p{}^o$ is the collection of all generalized open sets in X.

Theorem 3.3:

For any $A, B \subseteq X$, the following hold:

1. $A \subseteq$ FPF-Gcl(A)
2. FPF-Gcl(A) is a generalized closed set.
3. If $A \subseteq B$, then FPF-Gcl(A) ⊆ FPF-Gcl(B).
4. FPF-Gcl(A∪B) = FPF-Gcl(A)∪FPF-Gcl(B), but FPF-Gcl(A∩B)⊆FPF-Gcl(A) ∩ FPF-Gcl(B) in general.

Proof:

Since FPF-Gcl(A) is defined as the smallest generalized closed set containing A, it follows directly that A ⊆FPF-Gcl(A). By definition, FPF-Gcl(A)is formed by the intersection of generalized closed sets, which is itself generalized closed. If $A \subseteq B$, then every generalized closed set containing B must also contain A, leading to FPF-Gcl(A)⊆FPF-Gcl(B)

The first equality follows because any closed set containing A or B must contain their union. The second relation is an inclusion because intersections of closures may exclude limit points.

Generalized Closed Sets in FPFTS

Definition 3.4:

A subset A of an FPFTS (X, $\mathscr{F}_p$) is Fermatean Picture Fuzzy Generalized Closed (FPF-GC) if $FPFcl(A) \subseteq U$ whenever $A \subseteq U$ and U is an Fermatean Picture Fuzzy open (FPF-O) set.

Theorem 3.5: The intersection of any number of FPF-GC sets is FPF-GC set.

Proof: Let $\{A_i\}_{i\in I}$ be a collection of FPF-GC sets. Since each A_i satisfies $FPFcl(A_i) \subseteq U$ for any FPF-O set U, their intersection $\cap_{i\in I} A_i$ also satisfies this condition, proving the theorem.

Example 3.6: Consider a universe $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.85,0.1,0.05), (X, 0.8,0.15,0.05)\}, B = \{(X, 0.9,0.05,0.05), (X, 0.75,0.2,0.05)\}$ are FPF-GC sets, then their intersection is also FPF-GC set.

Theorem 3.7: The union of two FPF-GC sets need not be FPF-GC set.

Proof: Let A and B be two FPF-GC sets. If their union were necessarily FPF-GC, then $FPFcl(A \cup B) \subseteq U$ for any FPF-O set U, which is not always true. A counterexample exists where some points in $FPFcl(A \cup B)$ do not satisfy the closure condition, thus proving the theorem.

Example 3.8: Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.7,0.2,0.1), (X, 0.6,0.3,0.1)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ are FPF-GC sets, then their union $A \cup B$ may not be FPF-GC set.

Theorem 3.9: Every Fermatean Picture Fuzzy closed (FPF-C) set is an FPF-GC set, but the converse need not hold.

Proof: Since an FPF-C set satisfies $FPFcl(A) = A$, it follows that every FPF-C set is also FPF-GC set. However, there exist FPF-GC sets that do not meet the definition of full closedness, showing the converse is not necessarily true.

Example 3.10: Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p= \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05), (X, 0.85,0.1,0.05)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ then A is FPF-GC but not FPF-C set.

Generalized Pre-Closed Sets in FPFTS

Definition 3.11: A subset A of an FPFTS (X, $\mathscr{T}_p$) is Fermatean Picture Fuzzy Generalized Pre-Closed (FPF-GPC) if $FPFpcl(A) \subseteq U$ whenever $A \subseteq U$ and U is an FPF-O set.

Theorem 3.12: The intersection of any number of FPF-GPC sets is FPF-GPC set.

Proof: Let $\{A_i\}_{i \in I}$ be a collection of FPF-GPC sets. Since each A_i satisfies $FPFpcl(A_i) \subseteq U$ for any FPF-O set U, their intersection $\cap_{i \in I} A_i$ also satisfies this condition, proving the theorem.

Example 3.13: Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.8,0.15,0.05), (X, 0.75,0.2,0.05)\}$ and $B = \{(X, 0.85,0.1,0.05), (X, 0.9,0.05,0.05)\}$ are FPF-GPC, then their intersection $A \cap B$ is also FPF-GPC.

Theorem 3.14: Every FPF-C set is an FPF-GPC set, but the converse need not hold.

Proof: Since an FPF-C set satisfies $FPFpcl(A) = A$, it follows that every FPF-C set is also FPF-GPC set. However, there exist FPF-GPC sets that do not meet the definition of full closedness, showing the converse is not necessarily true.

Example 3.15: Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05), (X, 0.8,0.1,0.1)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ then A is FPF-GPC set but not FPF-C set.

Theorem 3.16: The union of two FPF-GPC sets need not be FPF-GPC set.

Proof: Let A and B be two FPF-GPC sets. If their union were necessarily FPF-GPC, then $FPFpcl(A \cup B) \subseteq U$ for any FPF-O set U, which is not always true.

Example 3.17: Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p= \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.85,0.1,0.05), (X, 0.75,0.2,0.05)\}$ and $B = \{(X, 0.9,0.05,0.05), (X, 0.8,0.1,0.1)\}$ are FPF-GPC, then their union $A \cup B$ may not be FPF-GPC.

Theorem 3.18:

Every FPF-GC set is an FPF-GPC set but not conversely.

Proof: Let A be a FPF-GC set in X and let $A \subseteq U$ and U is a FPF-O set in (X, $\mathscr{F}_p$). Since $FPFpcl(A) \subseteq FPFcl(A)$ and by hypothesis, $FPFpcl(A) \subseteq U$. Therefore A is a FPF-GPC set in X.

Example 3.19:

Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05), (X, 0.85,0.1,0.05)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ then B is FPF-GPC set but not FPF-GC set.

Theorem 3.20:

Every FPF-PC set is a FPF-GPC set but not conversely.

Proof: Let A be a FPF-PC set in X and let $A \subseteq U$ and U is a FPF-O set in (X, $\mathscr{F}_p$). By hypothesis, $FPFcl(FPFint(A)) \subseteq A$. This implies that $FPFpcl(A) = A \cup FPFcl(FPFint(A)) \subseteq A$. Therefore $FPFpcl(A) \subseteq U$. Hence A is a FPF-GPC set in X.

Example 3.21:

Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05), (X, 0.85,0.1,0.05)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ then B is FPF-GPC set but not FPF-PC set.

Generalized Semi-Closed Sets in FPFTS

Definition 3.22: A subset A of an FPFTS (X, $\mathscr{F}_p$) is Fermatean Picture Fuzzy Generalized Semi-Closed (FPF-GSC) if $FPFscl(A) \subseteq U$ whenever $A \subseteq U$ and U is an FPF-O set.

Theorem 3.23: The union of two FPF-GSC sets need not be FPF-GSC set.

Proof: Let A and B be two FPF-GSC sets. If their union were necessarily FPF-GSC sets, then $FPFscl(A \cup B) \subseteq U$ for any FPF-O set U, which is not always true.

Example 3.24: Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.65,0.3,0.05), (X, 0.7,0.2,0.1)\}$ and $B = \{(X, 0.8,0.15,0.05), (X, 0.6,0.35,0.05)\}$ are FPF-GSC, then their union $A \cup B$ may not be FPF-GSC.

Theorem 3.25:

FPF-SC set and FPF-GPC set are independent to each other.

Example 3.26:

Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05), (X, 0.85,0.1,0.05)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ then the FPFS A is a FPF-SC set but not a FPF-GPC set in X.

Also, the FPFS B is a FPF-GPC set but not a FPF-SC set in X.

Theorem 3.27:

FPF-GSC sets and FPF-GPC sets are independent to each other.

Example 3.28:

Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05),\ (X, 0.85,0.1,0.05)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ then the FPFS A is a FPF-GSC but not a FPF-GPC in X.

Also, the FPFS B is a FPF-GPC but not a FPFG-SC in X.

Generalized α-Closed Sets in FPFTS

Definition 3.29: A subset A of an FPFTS (X, $\mathscr{F}_p$) is Fermatean Picture Fuzzy Generalized α-Closed (FPF-GαC) if $FPF\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an FPF-O set.

Theorem 3.30: If A is an FPF-GαC set, then $FPFcl(FPFint(FPFcl(A))) \subseteq A$.

Proof: Since A satisfies the α-closure condition, its interior closure remains contained within A, reinforcing the inclusion $FPFcl(FPFint(FPFcl(A))) \subseteq A$.

Example 3.31: Let $X = \{a, b, c\}$ and $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.85,0.1,0.05), (X, 0.8,0.15,0.05)\}$, then it satisfies the condition for an FPF-GαC set.

Theorem 3.32: The intersection of any number of FPF-GαC sets is FPF-GαC set.

Proof: Let $\{A_i\}_{i\in I}$ be a collection of FPF-GαC sets. Since each A_i satisfies $FPF\alpha cl(Ai) \subseteq U$ for any FPF-O set U, their intersection $\cap_{i\in I} A_i$ also satisfies this condition, proving the theorem.

Example 3.33: Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.8,0.15,0.05), (X, 0.75,0.2,0.05)\}$ and $B = \{(X, 0.85,0.1,0.05), (X, 0.9,0.05,0.05)\}$ are FPF-GαC, then their intersection $A \cap B$ is also FPF-GαC.

Theorem 3.34: Every FPF-C set is an FPF-GαC set, but the converse need not hold.

Proof: Since an FPF-C set satisfies $FPF\alpha cl(A) = A$, it follows that every FPF-C set is also FPF-GαC. However, there exist FPF-GαC sets that do not meet the definition of full closedness, showing the converse is not necessarily true.

Example 3.35: Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05), (X, 0.8,0.1,0.1)\}$, then A is FPF-GαC set but not a FPF-C set.

Theorem 3.36: The union of two FPF-GαC sets need not be FPF-GαC set.

Proof: Let A and B be two FPF-GαC sets. If their union were necessarily FPF-GαC, then $FPF\alpha cl(A \cup B) \subseteq U$ for any FPF-open set U, which is not always true.

Example 3.37: Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.85,0.1,0.05), (X, 0.75,0.2,0.05)\}$ and $B = \{(X, 0.9,0.05,0.05), (X, 0.8,0.1,0.1)\}$ are FPF-GαC sets, then their union $A \cup B$ may not be FPF-GαC.

Theorem 3.38:

Every FPF-αC set is FPF-GαC set but not conversely.

Proof: Let A be a FPF-αC set in X and let $A \subseteq U$ and U is a FPF-αO set in (X, $\mathscr{F}_p$). We know that, $FPFcl(FPFint(A)) \subseteq A$. This implies $FPF\alpha cl(A) = A \cup FPFcl(FPFint(A)) \subseteq A$. Therefore, $FPF\alpha cl(A) \subseteq U$. Hence, A is a FPF-GαC set in X.

Example 3.39:

Let $X = \{a, b, c\}$ with topology $\mathscr{F}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9,0.05,0.05),\ (X, 0.85,0.1,0.05)\}$ and $B = \{(X, 0.8,0.1,0.1), (X, 0.9,0.05,0.05)\}$ then the FPFS B is FPF-GαC set but not a FPF-αC set in X.

Theorem 3.40:

Every FPF-GC set is FPF-GαC set but not conversely.

Proof: Let A be a FPF-GC set in X and let $A \subseteq U$ and U is a FPF-αO set in (X, $\mathscr{T}_p$). Since $FPF\alpha cl(A) \subseteq FPFcl(A)$ and by hypothesis**,** $FPF\alpha cl(A) \subseteq U$. Therefore, A is a FPF-GαC in X.

Example 3.41:

Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9, 0.05, 0.05), (X, 0.85, 0.1, 0.05)\}$ and $B = \{(X, 0.85, 0.05, 0.1), (X, 0.88, 0.07, 0.05)\}$ then the FPFS B is a FPF-GαC but not a FPF-GC set in X.

Theorem 3.42:

Every FPF-GαC set is FPF-GPC but not conversely.

Proof:

To Prove: If A is $FPF - G\alpha C \rightarrow A$ is $FPF - GPC$.

Given: $\alpha cl\,(A) = cl\,(int\,(cl\,(A)))$

We know $int\,(cl\,(A)) \subseteq int\,(A)$ (in fuzzy setting, inclusion holds weakly)

Hence, $\alpha cl\,(A) \subseteq cl\,(int\,(A)) = pcl\,(A)$

So if $\alpha cl\,(A) \subseteq U$, then $pcl\,(A) \subseteq U$, thus $FPF - GPC$

Then, $FPF - G\alpha C \subseteq FPF - GPC$.

Example 3.43:

Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9, 0.05, 0.05), (X, 0.85, 0.1, 0.05)\}$ and $B = \{(X, 0.8, 0.1, 0.1), (X, 0.9, 0.05, 0.05)\}$ then the FPFS B is a FPF-GPC set but not a FPF-GαC set in X.

Theorem 3.44:

Every FPF-GαC set is FPF-GSC but not conversely.

Proof:

Let $A \subseteq U$ and U is FPF-GαO.

We know that, $\alpha cl\,(A) \subseteq U$, this implies that $cl(int\,(cl\,(A))) \subseteq U$.

It means, $int\,(cl\,(A)) \subseteq U$ then $A \cup int\,(cl\,(A)) \subseteq U$.

Therefore, $scl\,(A) \subseteq U$, U is FPFO set.

Hence A is FPF-GSC.

Example 3.45:

Let $X = \{a, b, c\}$ with topology $\mathscr{T}_p = \{\emptyset, X, A, B\}$.

If $A = \{(X, 0.9, 0.1, 0.0), (X, 0.7, 0.2, 0.1)\}$ and $B = \{(X, 0.85, 0.1, 0.05), (X, 0.75, 0.2, 0.05)\}$ then the set $C = \{(X, 0.8, 0.1, 0.1), (X, 0.75, 0.2, 0.05)\}$ is a FPF-GSC set but not a FPF-GαC set in X.

Conclusion:

This study examined generalized open and closed sets in Fermatean Picture Fuzzy Topological Spaces (FPFTS), including generalized closed, pre-closed, semi-closed and α-closed sets. Their fundamental properties, theorems and proofs were analyzed alongside numerical examples to illustrate their significance.

The research findings demonstrate that FPFTS provides a broader and more flexible structure for handling uncertainty compared to classical fuzzy topology. The relationships between different types of generalized closed sets were explored.

Future research could extend these findings by applying FPFTS in real-world applications such as medical diagnosis, machine learning, and optimization problems. Additionally, further studies on multi-dimensional FPFTS and continuity concepts could provide new theoretical insights and practical applications in fuzzy topology.

References:

[1] Atanassov K., Intuitionistic Fuzzy sets, Fuzzy sets and systems, 20, pp. 87-96, 1986.

[2] Atanassov. K, Intuitionistic Fuzzy Sets, Theory and Applications, Heidelberg: Physica-Verlag, (1999), 1-137.

[3] Bhattacharya, P. "Fuzzy topology." Fuzzy Sets and Systems, 17(3), 239-248, (1985).

[4] Chang C., Fuzzy topological spaces, J. Math. Anal. Appl., 24, pp. 182-190, 1968,

[5] Coker D., An introduction to Fuzzy topological space, fuzzy sets and systems, 88, pp. 81-89, 1997.

[6] Hariwan Z. Ibrahim, Fermatean fuzzy topological spaces, J. Appl. Math. And Informatics, 40, 85-98, 2022.

[7] Kadambavanam. K and Vaithiyalingam.K., Weakly π Generalized Closed Sets in Intuitionistic Fuzzy Topological Spaces and its Applications, International Journal of Modern Sciences and Engineering Technology (IJMSET), 3(1), 45-52, 2016.

[8] M. Udhaya Shalini, Dr. A. Stanis Arul Mary, Generalized Pre–closed Sets in Pythagorean Fuzzy Topological Spaces, IJCRT ,Volume 10, Issue 3 March 2022.

[9] Rajarajeswari P., Senthil Kumar L., Generalized Pre-closed Sets in Intuitionistic Fuzzy Topological Spaces, International Journal of Fuzzy Mathematical and Systems, Volume 1, Number 3 (2011), pp. 253-262.

[10] Senapati. T and Yager. R. R, Fermatean fuzzy sets, Journal of Ambient Intelligence and Humanized Computing, 11,663 - 674, 2020.

[11] Sostak, A. "On a fuzzy topological structure." Mathematical Slovaca, 31(2), 217-232, (1981).

[12] S.S. Thakur and Rekha Chaturvedi, Generalized closed sets in intuitionistic fuzzy topology, The journal of Fuzzy Mathematics, 16, 559-572, 2008.

[13] S. S. Thakur and Rekha Chaturvedi.,Regular generalized closed sets in intuitionistic fuzzy topological spaces, Universitatea Din Bacau StudiiSi Cercertar Stiintifice, 6, 257-272, 2006.

[14] Thakur S.S., and Rekha Chaturvedi., Regular generalized closed sets in Intuitionistic Fuzzy topological Spaces, Universitatea Din Bacau, Studii Si Cercetari Stiintifice, Seria: Mathematica, 16, pp. 257-272, 2006.

[15] V.Chitra, B.Maheswari, Fermatean Picture Fuzzy Sets, Advances in Nonlinear Variational Inequalities, 28, 390-399, 2024.

[16] V.Chitra, B.Maheswari, S. Jafari., Fermatean Picture fuzzy continuity in Fermatean Picture fuzzy topological spaces, Communications in Mathematics and Applications, Communicated.

[17] V.Chitra, B.Maheswari, Fermatean Picture fuzzy connectedness and its compactness, International Journal of Mathematics Trends and Technology, Volume 71 Issue 2, 7-14, February 2025. https://doi.org/10.14445/22315373/IJMTT-V71I2P102.

[18] Yager, R.R. "Pythagorean fuzzy subsets." IEEE Transactions on Fuzzy Systems, 22(4), 947-965, (2013).

[19] Zadeh L. A., Fuzzy sets, Information and control, 8, pp. 338-353, 1965.