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Some Of The P_3 Colorable Graphs

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Abstract

Vertex coloring is a well-known graph coloring method. This article introduces a new graph coloring called P_3 coloring. A graph is considered P_3 colorable if colors can be assigned to its vertices, such that each P_3 path contains distinct vertices. The aim of this article is to extend and prove some basic results and compute P_3 chromatic numbers for various graphs such as Möbius-Kantor, Sun, Centipede, Franklin and Triangular Ladder graphs.

Keywords: Graph coloring; Chromatic number; Möbius –Kantor graph; Sun graph; Centipede graph; Franklin graph; Triangular Ladder graph

Introduction

Graph coloring is an essential concept in graph theory, widely explored for both its theoretical depth and practical applications (West, 2017[17]; Bondy & Murty, 2008)[2]) Traditionally, vertex coloring involves assigning colors to the vertices of a graph in such a way that no two adjacent vertices share the same color (Chartrand & Zhang, 2009)[3]). Over time, this idea has evolved into more specialized techniques, such as list coloring (Baber, 2005)[1], acyclic coloring (Fertin et al., 2004)[6]), and total coloring (Isobe, 2008)[8]. A more recent development in this field is P_3-coloring, which has garnered increasing interest in mathematical research (Naeem et al., 2023)[11] . A graph is P_3 colorable if colors can be assigned to its vertices so that every three-vertex path (P_3) consists of distinct colors. This introduces new combinatorial challenges and extends the traditional chromatic number concept. Graph coloring has many real-world applications, from scheduling and resource allocation (Reed, 2001[13]; Garey & Johnson, 2003[7]),to frequency assignment and register allocation in computer science (Tovey, 2002 [16]; Skiena, 2020 [14]). Researchers have also explored related techniques, such as circular and equitable colorings, to optimize network structures and solve combinatorial problems (Nogueira & Skrekovski, 2015[12]; Xu & Wu, 2018 [19]). In this study, we focus on P_3 coloring for well-known graph families, including the Möbius-Kantor graph, Sun graph, Centipede graph, Queen Graph and Triangular Ladder graph. By determining their P_3 chromatic numbers, we establish key results that contribute to the broader study of graph coloring. Our findings build on previous research on symmetric graph structures and lay the groundwork for further exploration in combinatorial graph theory (Diestel, 2017[5]; Thomassen, 2016 [15]).

Ultimately, this research aims to analyze the P_3-chromatic properties of various graphs and highlight their significance in graph theory. By understanding how different graphs behave under P_3-coloring constraints, we aim to, motivated by the above reasoning, we introduce the P_3 labeling of graphs, and we will discuss this labeling for some very well-known graphs.

Preliminaries

P_3 coloring of graph

A P_3 coloring is a function g from the vertex set of graph G to the set of colors $\{C_1, C_2, C_3, \ldots, C_k\}$ such that for every P_3 path on graph G, the colors of its vertices are distinct, that is, if xyz is a P_3 path on G, then $g(x) \neq g(y) \neq g(z) \neq g(x)$.

Chromatic Number

The minimum number of colors required for a graph G to have P_3 coloring is called the P_3 chromatic number, and it is denoted as χ (G). Note that every P_3 coloring of a graph is also its P_3 coloring. Therefore, we have

$$\chi(G) \leq \chi_3(G).$$

In addition, it is clear from the definition that for all graphs G, $\chi_3(G) \geq 3$.

Results

In this section, we present some results based on the P_3 coloring in graphs.

Lemma 1 [11]

Let G be a graph with a vertex that is adjacent to every other vertex in G. Then, the P_3-chromatic number of G, denoted as $\chi_3(G)$, equals the total number of vertices in G, i.e., $\chi_3(G) = n$.

From Lemma 1, we can derive the following corollaries:

Corollary 1: If J_n represents the Jellyfish graph with n vertices, then its P_3-chromatic number satisfies

$\chi_3(J_n) = n$ for all n.

Corollary 2: If S_n denotes the Sunflower graph with n vertices, then $\chi_3(S_n) = n$ for all n.

Theorem 1

Let M_n be the Möbius-Kantor graph and its P_3-chromatic number is always 4 for all n,

i.e., $\chi_3(M_n) = 4$

Proof:

Consider the Möbius-Kantor graph M_n on n vertices, where $n \geq 3$, as illustrated in Figure 1. To establish that M_n admits a valid P_3 coloring, we define a function g: $V(M_n) \rightarrow \{1, 2, 3, 4\}$ as follows:

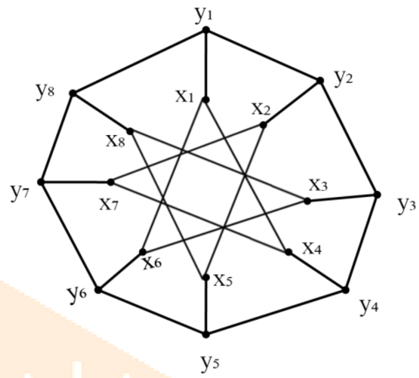

Figure: 1. Möbius-Kantor graph of n vertices

$$g(x_i) = \begin{cases} 1, \text{ if } i \equiv 1 \pmod 4, 1 \leq i \leq 8; \\ 2, \text{ if } i \equiv 2 \pmod 4, 1 \leq i \leq 8; \\ 3, \text{ if } i \equiv 3 \pmod 4, 1 \leq i \leq 8; \\ 4, \text{ if } i \equiv 4 \pmod 4, 1 \leq i \leq 8. \end{cases}$$

$$g(y_i) = \begin{cases} 3, \text{ if } i \equiv 0 \pmod 4, 1 \leq i \leq 8; \\ 4, \text{ if } i \equiv 1 \pmod 4, 1 \leq i \leq 8; \\ 1, \text{ if } i \equiv 2 \pmod 4, 1 \leq i \leq 8; \\ 2, \text{ if } i \equiv 3 \pmod 4, 1 \leq i \leq 8. \end{cases}$$

To verify that g is a valid P_3 coloring of M_n, we examine all possible P_3 paths in the graph such as $x_1y_1y_2$, $x_1x_4x_6$, $x_1x_4y_4$, $x_2x_3x_4$, $y_1y_2y_3$, $y_4y_5x_5$ and verify that each path consists of three distinct colors.

(a). If $i \equiv 0 \pmod 4$, then $g(x_i) = 4$, $g(x_{i+1}) = 3$,$g(x_{i+2}) = 2$;

(b). If $i \equiv 1 \pmod 4$, then $g(x_i) = 1$, $g(x_{i+1}) = 2$, $g(x_{i+2}) = 3$;

(c). If $i \equiv 2 \pmod 4$, then $g(x_i) = 2$, $g(x_{i+1}) = 3$, $g(x_{i+2}) = 4$.

(d). If $i \equiv 3 \pmod 4$, then $g(x_i) = 3$ $g(x_{i+1}) = 4$ $g(x_{i+2}) = 1$.

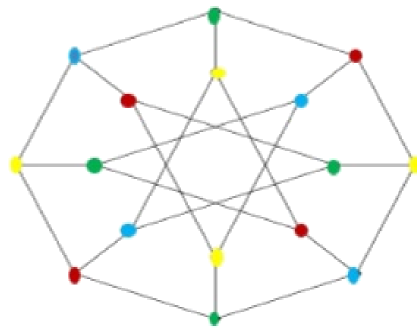

Figure: 2. P_3 coloring of Mn.

From the cases above, it is evident (figure 2) that all possible P_3 paths within M_n contain three distinct colors under the labeling function g. Thus, g is a valid P_3 coloring. Therefore, we conclude that $\chi_3(M_n) = 4$.

Theorem: 2. Let Sn be the Sun graph and $n \neq 5$. Then, for all $n \geq 3$

$$\chi_3(Sn) = \begin{cases} 6, & \text{if } i \equiv 0 \pmod 3; \\ 7, & \text{if } i \equiv 1 \pmod 3; \\ 7, & \text{if } i \equiv 2 \pmod 3) \end{cases}$$

Proof:

Let Sn be a sun graph on n vertices, this proof comprises of three cases on three different values of n under mod 3**.**

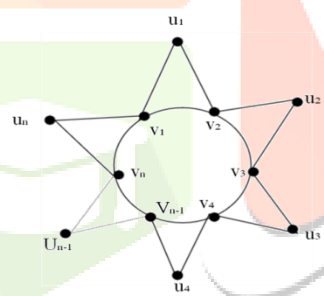

Figure: 1. Sun graph of n vertices

Case I: Assume that **$n \equiv 0 \pmod 3$**.

Let us define a function f: Sn →{1, 2, 3, 4, 5, 6}, as follows:

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod 3; \\ 2, & \text{if } i \equiv 2 \pmod 3; \\ 3, & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

$$f(u_i) = \begin{cases} 4, & \text{if } i \equiv 1 \pmod 3; \\ 5, & \text{if } i \equiv 2 \pmod 3; \\ 6, & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

where $1 \leq i \leq n - 4$. It illustrates the P_3 coloring of Sn to show this labeling. Let Q_1: $v_1v_{i+1}u_{i+2}$ be an arbitrary P_3 path in Sn,

Then there are some three possible cases.

(a). If $i \equiv 0 \pmod 3$, then Q_1 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(v_i) = 3$, $f(v_{i+1}) = 1$, $f(u_{i+1}) = 4$;
(ii) If the P_3 path $v_iv_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_i) = 3$, $f(v_{i+1}) = 1$, $f(v_{i+2}) = 2$;
(iii) If the P_3 path $v_iu_iv_{i+1}$, then the P_3 path colors are $f(v_i) = 3$, $f(u_i) = 6$, $f(v_{i+1}) = 1$;
(iv) If the P_3 path $v_{i+1}u_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 1$, $f(u_{i+1}) = 4$, $f(v_{i+2}) = 2$;
(v) If the P_3 path $u_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(u_i) = 6$, $f(v_{i+1}) = 1$, $f(u_{i+1}) = 5$;
(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 4$, $f(v_{i+2}) = 2$, $f(u_{i+2}) = 5$;

(b). If $i \equiv 1 \pmod 3$, then Q_1 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$;
(ii) If the P_3 path $v_iv_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(v_{i+2}) = 3$;
(iii) If the P_3 path $v_iu_iv_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(u_i) = 4$, $f(v_{i+1}) = 2$;
(iv) If the P_3 path $v_{i+1}u_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$;
(v) If the P_3 path $u_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(u_i) = 4$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 6$;
(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$, $f(u_{i+2}) = 6$;

(c). If $i \equiv 2 \pmod 3$, then Q_1 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(v_i) = 2$, $f(v_{i+1}) = 3$, $f(u_{i+1}) = 6$;
(ii) If the P_3 path $v_iv_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_i) = 2$, $f(v_{i+1}) = 3$, $f(v_{i+2}) = 1$;
(iii) If the P_3 path $v_iu_iv_{i+1}$, then the P_3 path colors are $f(v_i) = 2$, $f(u_i) = 5$, $f(v_{i+1}) = 3$;
(iv) If the P_3 path $v_{i+1}u_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 3$, $f(u_{i+1}) = 6$, $f(v_{i+2}) = 1$;
(v) If the P_3 path $u_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(u_i) = 5$, $f(v_{i+1}) = 3$, $f(u_{i+1}) = 6$;
(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 6$, $f(v_{i+2}) = 1$, $f(u_{i+2}) = 4$;

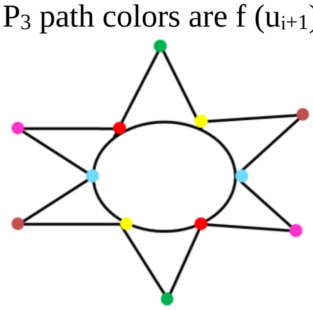

Figure: 2. P_3 coloring of S_6

So, in all of these cases, it is clear that the graphs corresponding to P_3 paths have vertices of three different colors. Therefore, f is a P_3 coloring, hence the result follows.

Case II: Suppose that $\mathbf{n \equiv 1 \pmod 3}$.

In this instance, the final vertex vn cannot be given any of the three colors provided when we begin coloring Sn from any vertex, such as v_1, to the last vn, using a maximum of three different colors. Therefore, for this graph to have a P_3 coloring, at least four colors are required. The P_3-labeling function is defined as follows:

$f(v_n) = 1$, $f(v_{n-1}) = 2$, $f(v_{n-2}) = 3$, $f(v_{n-3}) = 4$ and for all $1 \leq i \leq n-4$ and

$f(u_n) = 5$, for all $1 \leq i \leq n-4$

we have

$$f(v_i) = \begin{cases} 4, & \text{if } i \equiv 1 \pmod 3; \\ 3, & \text{if } i \equiv 2 \pmod 3; \\ 2, & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

$$f(u_i) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 3; \\ 6, & \text{if } i \equiv 1 \pmod 3; \\ 7, & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Let Q_2 be an arbitrary path in Sn; then, there are possible cases to discuss this labeling

(a). If $i \equiv 0 \pmod 3$, then Q_2 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 4$;
(ii) If the P_3 path $v_iv_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(v_{i+2}) = 3$;
(iii) If the P_3 path $v_iu_iv_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(u_i) = 4$, $f(v_{i+1}) = 2$;

(iv) If the P_3 path $v_{i+1}u_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$;

(v) If the P_3 path $u_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(u_i) = 4$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 6$;

(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$, $f(u_{i+2}) = 6$;

(b). If $i \equiv 1 \pmod 3$, then Q_2 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 4$;
(ii) If the P_3 path $v_iv_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(v_{i+2}) = 3$;
(iii) If the P_3 path $v_iu_iv_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(u_i) = 4$, $f(v_{i+1}) = 2$;

(iv) If the P_3 path $v_{i+1}u_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$;

(v) If the P_3 path $u_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(u_i) = 4$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 6$;

(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$, $f(u_{i+2}) = 6$;

(c). If $i \equiv 2 \pmod 3$, then Q_1 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 4$;
(ii) If the P_3 path $v_iv_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(v_{i+2}) = 3$;
(iii) If the P_3 path $v_iu_iv_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(u_i) = 4$, $f(v_{i+1}) = 2$;

(iv) If the P_3 path $v_{i+1}u_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$;

(v) If the P_3 path $u_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(u_i) = 4$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 6$;

(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$, $f(u_{i+2}) = 6$;

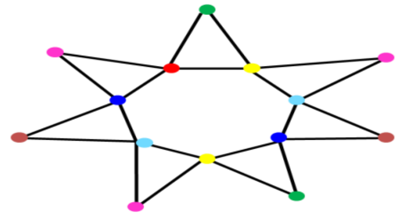

Figure: 3. P_3 coloring of S_6

Case III: When $\mathbf{n \equiv 2 (mod\ 3)}$.

In this instance, the final two vertices, vn-1, vn, and u_1 to un, cannot be given any of the three colors provided in order to begin coloring Sn from vertex v_1 to the vn with a maximum of three colors. As a result, for this graph to have P3 coloring, at least four colors are required. The P_3 color labeling function will be defined as follows for the forward case:

$$f(v_1) = 1,\ f(v_2) = 2,\ f(v_3) = 3,\ f(v_4) = 4,\ f(v_n) = 4,\ f(v_{n-1}) = 3,\ f(v_{n-2}) = 2,$$

$$f(v_{n-3}) = 1,\ \forall\ 5 \leq i \leq n-4 \text{ and}$$

the function is defined by

$$f(v_i) = \begin{cases} 2, & \text{if } i \equiv 2 \pmod 3; \\ 3, & \text{if } i \equiv 0 \pmod 3; \\ 4, & \text{if } i \equiv 1 \pmod 3. \end{cases}$$

$$f(u_i) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 3; \\ 6, & \text{if } i \equiv 1 \pmod 3; \\ 7, & \text{if } i \equiv 2 \pmod 3) \end{cases}$$

(**a**). If $i \equiv 0 \pmod 3$ and $5 \leq i \leq n-4$, then Q_1 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_i v_{i+1} u_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 4$;
(ii) If the P_3 path $v_i v_{i+1} v_{i+2}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(v_{i+2}) = 3$;
(iii) If the P_3 path $v_i u_i v_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(u_i) = 4$, $f(v_{i+1}) = 2$;
(iv) If the P_3 path $v_{i+1} u_{i+1} v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$;
(v) If the P_3 path $u_i v_{i+1} u_{i+1}$, then the P_3 path colors are $f(u_i) = 4$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 6$;
(vi) If the P_3 path $u_{i+1} v_{i+2} u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$, $f(u_{i+2}) = 6$;

(b). If $i \equiv 1 \pmod 3$ and $5 \leq i \leq n-4$, then Q_1 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_i v_{i+1} u_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 4$;
(ii) If the P_3 path $v_i v_{i+1} v_{i+2}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(v_{i+2}) = 3$;
(iii) If the P_3 path $v_i u_i v_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(u_i) = 4$, $f(v_{i+1}) = 2$;
(iv) If the P_3 path $v_{i+1} u_{i+1} v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$;
(v) If the P_3 path $u_i v_{i+1} u_{i+1}$, then the P_3 path colors are $f(u_i) = 4$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 6$;

(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$, $f(u_{i+2}) = 6$;

(c). If $i \equiv 2 \pmod 3$ and $5 \le i \le n - 4$, then Q_1 be an arbitrary P_3 path in Sn and we have some possible P_3 colorings as follows,

(i) If the P_3 path $v_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 4$;
(ii) If the P_3 path $v_iv_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_i) = 1$, $f(v_{i+1}) = 2$, $f(v_{i+2}) = 3$;
(iii) If the P_3 path $v_iu_iv_{i+1}$, then the P_3 path colors are $f(v_i) = 1$, $f(u_i) = 4$, $f(v_{i+1}) = 2$;

(iv) If the P_3 path $v_{i+1}u_{i+1}v_{i+2}$, then the P_3 path colors are $f(v_{i+1}) = 2$, $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$;

(v) If the P_3 path $u_iv_{i+1}u_{i+1}$, then the P_3 path colors are $f(u_i) = 4$, $f(v_{i+1}) = 2$, $f(u_{i+1}) = 6$;

(vi) If the P_3 path $u_{i+1}v_{i+2}u_{i+2}$, then the P_3 path colors are $f(u_{i+1}) = 5$, $f(v_{i+2}) = 3$, $f(u_{i+2}) = 6$;

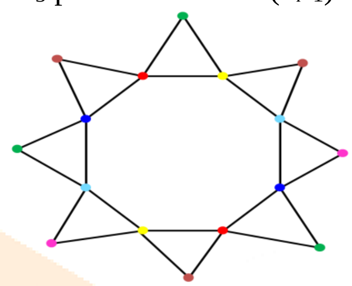

Figure: 4. P_3 coloring of S_8

The P_3 coloring of Sn is thus evident from all of these conditions; the vertices of every P_3 path have a variety of colors. Consequently, f has a P_3 coloring, and $\chi_3(Sn) < 7$ ∀ $n \ge 8$. The proof is thus finished. □

Theorem: 3. Let Cn be the centipede graph for all $n \ge 3$. Then $\chi_3(Cn) = 5$

Proof:

Let Cn be the centipede graph with $n \ge 3$. It is formed by attaching a single pendant vertex v; to each vertex u; of a path $u_1, u_2 \ldots U_n$, where $n > 3$. From earlier results, we know that $\chi_3(Cn) = 5$. We now construct an explicit labeling to demonstrate that five colors are sufficient for a valid P_3 coloring.

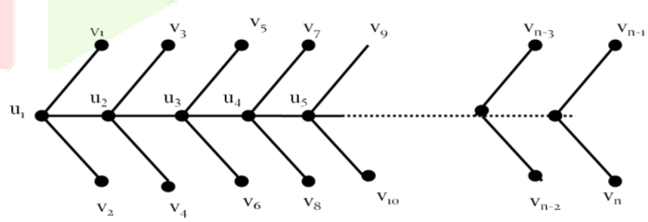

Figure: 1. A Centipede graph Cn with n vertices

Define a coloring function

$$V(Cn) \rightarrow \{1, 2, 3, 4, 5\}$$

as follows:

$$g(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2,\ 1 \le i \le n; \\ 2, & \text{if } i \equiv 1 \pmod 2,\ 1 \le i \le n. \end{cases}$$

$$g(u_i) = \begin{cases} 3, \text{ if } i \equiv 1 \pmod 3,\ 1 \le i \le n; \\ 4, \text{ if } i \equiv 2 \pmod 3,\ 1 \le i \le n; \\ 5, \text{ if } i \equiv 0 \pmod 3,\ 1 \le i \le n. \end{cases}$$

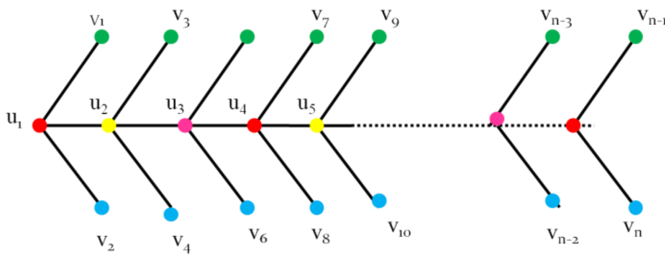

Figure: 2. A P_3 coloring of C_8

This ensures every vertex receives exactly one color from the set {1, 2, 3, 4, 5}, and the assignments are based on simple modular conditions. Now, consider all possible subgraphs of C_n that are paths on three vertices Let Q be any arbitrary path of Cn; then, there are five possible types of P_3 paths in C_n. The paths are $v_iu_iu_{i+1}$, $v_{i+1}u_{i+1}\,u_{i+2}$, $v_{i+2}u_i\,u_{i+1}$, $u_i\,u_{i+1}\,u_{i+2}$, $u_{i+1}u_{i+2}u_{i+3}$.Thus, g is a valid P_3 coloring of C_n using five colors. Since five colors are sufficient and prior results confirm that at least five colors are necessary Therefore, we conclude that $\chi_3\,(C_n) = 5$.Thus, the proof is completed.□

Theorem: 4. Let TLn be the Triangular Ladder graph ∀ n ≥ 3. Then the χ_3 (TLn) ≤ 6

Proof:

Let TL_n be the triangular ladder graph with n ≥ 3. According to previously established results, we have

χ_3 (TLn) ≤ 6. In this proof, we construct an explicit vertex labeling to demonstrate that six colors are sufficient for a valid P_3 coloring of TL_n.

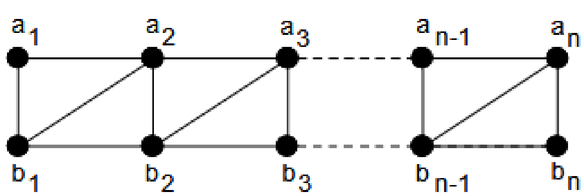

Figure: 1. A triangle ladder graph TLn with vertices

We define a labeling function

$$g: V(TLn) \rightarrow \{1, 2, 3, 4, 5, 6\}$$

as follows:

1, if $i \equiv 1 \pmod 3$, $1 \le i \le n$;

$$g(a_i) = \quad 2, \text{ if } i \equiv 2 \pmod 3,\ 1 \le i \le n;$$

$$3, \text{ if } i \equiv 0 \pmod 3,\ 1 \le i \le n.$$

$$g(b_i) = \begin{cases} 4, & \text{if } i \equiv 1 \bmod 4,\ 1 \le i \le n; \\ 5, & \text{if } i \equiv 2 \bmod 4,\ 1 \le i \le n; \\ 6, & \text{if } i \equiv 0 \bmod 4,\ 1 \le i \le n. \end{cases}$$

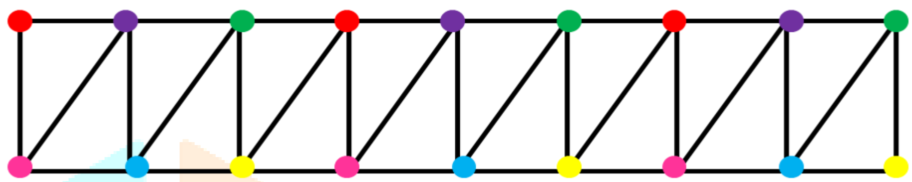

Figure: 2. A P_3 coloring of TL_8

For the reader, this labeling is explained in Figure 15. Let Q be any arbitrary path of TLn; then, there are ten possible types of P_3 paths in TLn, and they are as follows. The paths are $a_ia_{i+1}a_{i+2}$, $a_ia_{i+1}b_{i+1}$, $a_ib_ib_{i+1}$, $b_ib_{i+1}b_{i+2}$, $b_ia_{i+1}a_{i+2}$, $b_ib_{i+1}a_{i+2}$, $a_{i+1}a_{i+2}b_{i+2}$, $b_{i+1}a_{i+1}a_{i+2}$, $a_{i+2}b_{i+2}b_{i+1}$, and $b_{i+2}b_{i+1}a_{i+2}$. Thus, g is a valid P_3 coloring of TLn using six colors, and we conclude that: χ_3 (TLn) ≤ 6.

Theorem: 5.

Let F_n be the Franklin graph, then $\chi_3(F_n) = 6$ for all n =12.

Proof:

Consider the Franklin graph F_n on n vertices, where $n \ge 3$, as illustrated in Figure. To establish that F_n is a valid P_3 coloring, we define a function g: V $(F_n) \to \{1, 2, 3, 4,5, 6\}$ as follows:

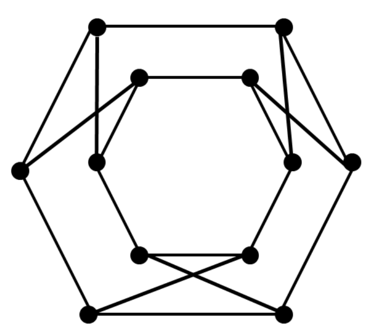

Figure: 1 Franklin graph n vertices

To verify that g is a valid P_3 coloring of F_n, we examine all possible P_3 paths in the graph and to prove that f is indeed a P_3 coloring in F_n, we show that all P_3 paths in Fn are of three different colors. Similarly, the theorem 1 follows the cases,

$$g(x_i) = \begin{cases} 1, \text{ if } i \equiv 0 \pmod 6; \\ 2, \text{ if } i \equiv 1 \pmod 6; \\ 3, \text{ if } i \equiv 2 \pmod 6: \\ 4, \text{ if } i \equiv 3 \pmod 6; \\ 5, \text{ if } i \equiv 4 \pmod 6; \\ 6, \text{ if } i \equiv 5 \pmod 6; \end{cases}$$

$$g(y_i) = \begin{cases} 2, \text{ if } i \equiv 0 \pmod 6; \\ 5, \text{ if } i \equiv 1 \pmod 6; \\ 4, \text{ if } i \equiv 2 \pmod 6: \\ 1, \text{ if } i \equiv 3 \pmod 6; \\ 6 \text{ if } i \equiv 4 \pmod 6; \\ 3, \text{ if } i \equiv 5 \pmod 6. \end{cases}$$

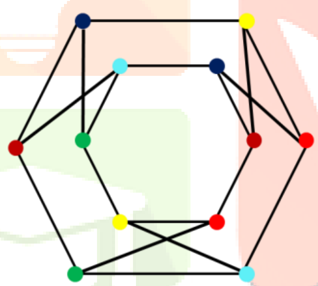

Figure: 2. P_3 coloring of F_n.

From the cases above, it is evident above graph that all possible P_3 paths within F_n contain three distinct colors under the labeling function g. Thus, g is a valid P_3 coloring. Therefore, we conclude that $\chi_3 (F_n) = 6$.

Conclusion

In this article, we presented a novel type of graph coloring method, which we referred to as P_3 coloring. We demonstrated that when a graph features a vertex that is adjacent to every other vertex in the graph, its P_3 chromatic number equals the total number of vertices in the graph. Subsequently, we compute the P_3 chromatic number for several well-known graphs. For this purpose, we determined the P_3 chromatic number for the Möbius–Kantor Graph, the Sun graph, a centipede graph, Franklin Graph and Triangular Ladder graph P_3 chromatic number of a graph is at least as large as or equal its chromatic number. The preceding discussion and findings outline some future directions, enhance progress in this area and broaden its application.

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