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A Brief Study Of A Special Type Of Set Named Midpoint Convex Set

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Abstract

This study investigates midpoint convex sets as a relaxation of classical convexity in linear spaces. Every convex set is midpoint convex, but the converse need not hold. We establish foundational results: stability under intersections, Minkowski combinations, linear images and pre-images, and translations. We also clarify when midpoint convexity upgrades to full convexity, notably under closedness, local boundedness, or mild regularity (e.g. measurability). Several examples and counterexamples are provided, and we correct a common misconception: unions of midpoint convex sets are generally not midpoint convex unless the family is nested (i.e. a chain). Short geometric figures illustrate the midpoint property, non-midpoint-convex union, and dyadic construction that underlies many proofs. These observations are relevant to functional analysis, convex geometry, optimisation, and the study of Jensen-type structures.

Keywords: Convex set, midpoint convex set, linear space, linear transformation, translation Dyadic convexity Jensen convexity.

1. Introduction

Convexity plays a fundamental role in functional analysis, optimisation, and topological studies. A subset A of a linear space L is **convex** if, for any two points $x, y \in A$, the line segment joining them lies entirely in A. That is,

$$A \text{ is convex } \Leftrightarrow \alpha x + (1-\alpha) y \in A \text{ for all } x, y \in A \text{ and } 0 \leq \alpha \leq 1.$$

This definition ensures that every convex combination of two points in the set remains within the set [1].

However, a weaker form of convexity can be defined by considering only the **midpoint** between two points rather than all convex combinations. This leads to the concept of a **midpoint convex set**, which relaxes the usual convexity condition but retains several useful geometric properties.

Let L be a real linear space. A set $A \subseteq L$ is convex if $\alpha x + (1-\alpha)y \in A$ for all $x, y \in A$ and all $\alpha \in [0,1]$ [1,3]. A weaker requirement is the midpoint condition, which only demands the membership of the midpoint:

Definition 1.1 Midpoint convex set

Let A set $A \subseteq L$ be midpoint convex if $x + y/2 \in A$ whenever $x, y \in A$ [2,4].

Every convex set is midpoint convex (choose $\alpha = 1/2$). The converse fails in general; nonetheless, with closedness in a topological vector space (TVS) or with measurability/local boundedness, midpoint convexity often implies convexity (see [1,3,5,6]). Figure 1 shows the midpoint condition.

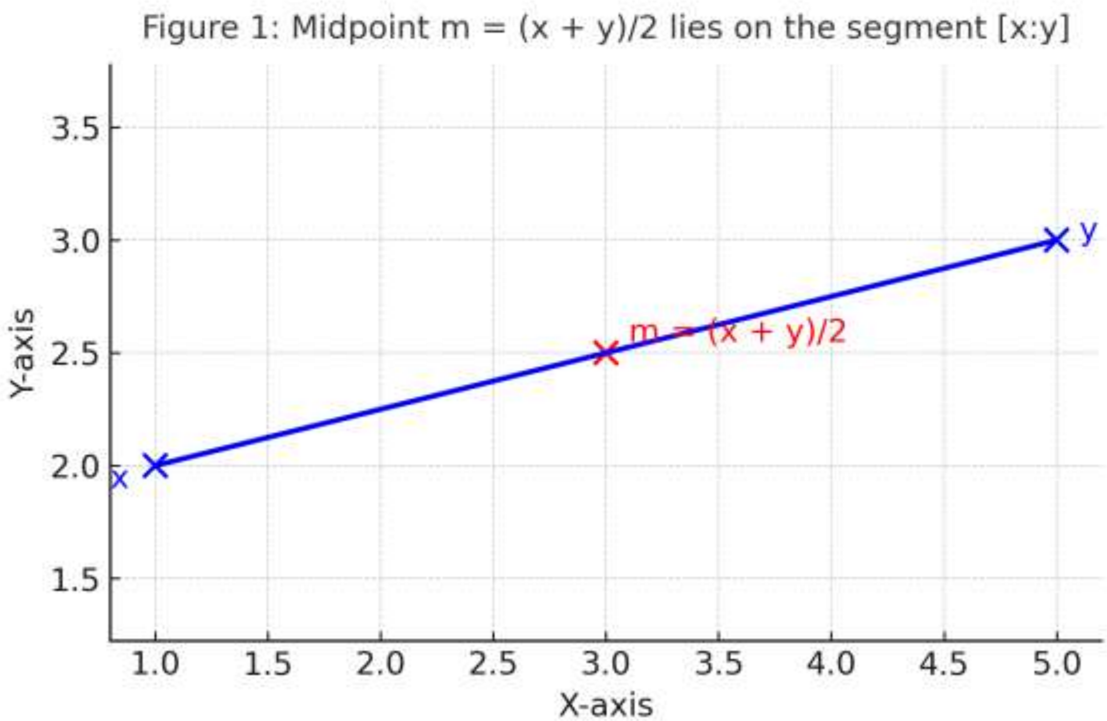

Figure 1. Midpoint $m = x + y/2$ lies on the segment $[x{:}\,y]$ (definition of midpoint convexity).

2. Preliminaries and Notation

For $x, y \in L$, denote the line segment by

$$[x{:}\,y] = \{\alpha x + (1-\alpha)y{:}\ \alpha \in [0,1]\}.$$

For a scalar λ and set $A \subseteq L$, let $\lambda A = \{\lambda a{:}\, a \in A\}$ and $A + B = \{a + b{:}\, a \in A,\ b \in B\}$. For $x \in L$, the translation is $x + A = \{x + a{:}\, a \in A\}$ [1,4].

3. Basic Properties and Theorems

Theorem 3.1 Midpoint characterization

If $A \subseteq L$ is midpoint convex, then

$$A = \left\{\frac{a_1 + a_2}{2} : a_1, a_2 \in A\right\}.$$

Proof. Trivial containment $A \subseteq \{\cdots\}$ uses $a = a + a/2$. The reverse follows from the midpoint property of the triangle.

Proof.
Let $a \in A$. Then clearly,

$$a = \frac{a + a}{2},$$

which implies $a \in \left\{\frac{a_1+a_2}{2} : a_1, a_2 \in A\right\}$. Hence,

$$A \subseteq \left\{\frac{a_1 + a_2}{2} : a_1, a_2 \in A\right\}.$$

Conversely, let $x = a_1 + a_2/2$ for some $a_1, a_2 \in A$.
Since A is midpoint convex, $x \in A$. Hence,

$$\left\{\frac{a_1 + a_2}{2} : a_1, a_2 \in A\right\} \subseteq A.$$

From (3.1) and (3.2), the following equality holds

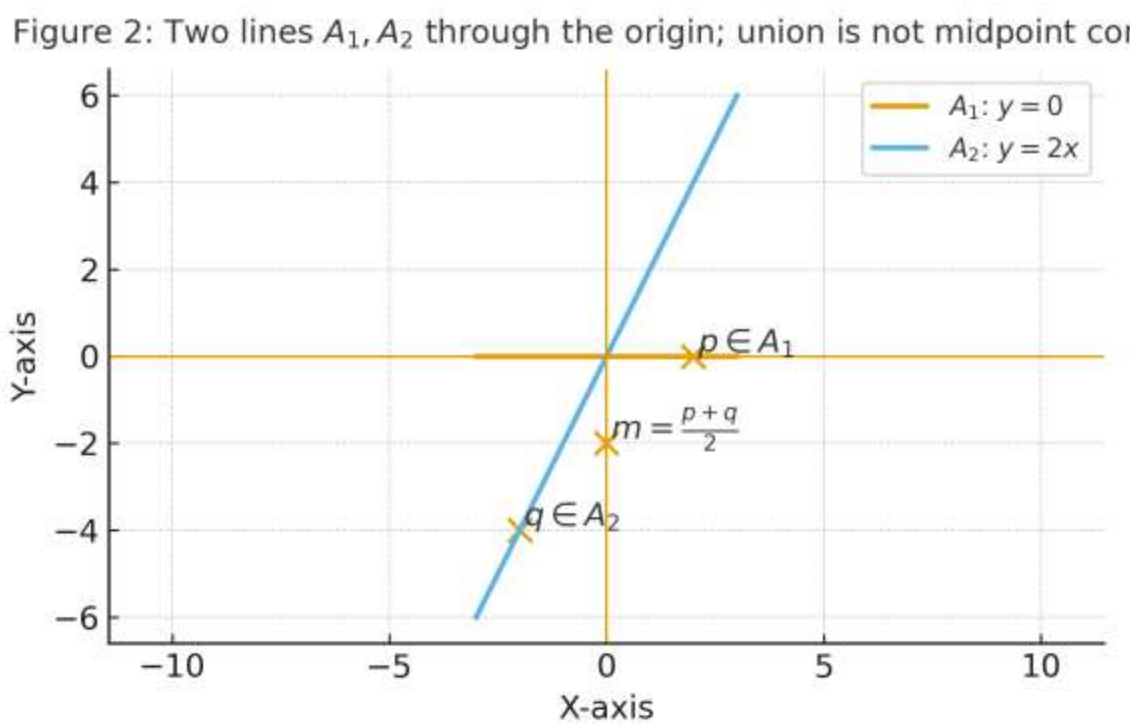

Figure 2. Two lines A_1, A_2 through the origin: both midpoint convex, but the union is not midpoint convex (midpoint of $p \in A_1$, $q \in A_2$ falls outside).

Theorem 3.2 (Minkowski combination with scalars).

If $A_1, A_2 \subseteq L$ are midpoint convex and $l_1, l_2 \in \mathbb{R}$, then

$$l_1 A_1 + l_2 A_2 = \{\, l_1 a_1 + l_2 a_2 : \ a_1 \in A_1,\ a_2 \in A_2 \,\}$$

is midpoint convex.
Proof.

Proof.
Let $x_1, x_2 \in l_1A_1 + l_2A_2$. Then
$x_1 = l_1a_1 + l_2b_1$ and $x_2 = l_1a_2 + l_2b_2$, where $a_1, a_2 \in A_1$ and $b_1, b_2 \in A_2$.
Now,

$$\frac{x_1 + x_2}{2} = l_1\left(\frac{a_1 + a_2}{2}\right) + l_2\left(\frac{b_1 + b_2}{2}\right).$$

Since A_1 and A_2 are midpoint convex, $a_1 + a_2/2 \in A_1$ and $b_1 + b_2/2 \in A_2$.
Hence, $x_1 + x_2/2 \in l_1A_1 + l_2A_2$. Therefore, the sum is midpoint convex.

Theorem 3.3 Intersection stability

The intersection of any family of midpoint-convex sets is also midpoint-convex.

Proof.
Let $\{A_i: i \in I\}$ be a family of midpoint convex sets in a linear space L, and define

$$A = \bigcap_{i \in I} A_i.$$

For any $x, y \in A$, we have $x, y \in A_i$ for each i.
Since each A_i is midpoint convex, $x + y/2 \in A_i$ for all i.
Hence, $x + y/2 \in \bigcap_{i \in I} A_i = A$.
Therefore, A is midpoint convex.

Remark (Unions are delicate).
The union of midpoint convex sets need **not** be midpoint-convex, even if they intersect pairwise.
Counterexample in $\mathbb{R}^2$: Let $A_1 = \{(t, 0): t \in \mathbb{R}\}$ and $A_2 = \{(t, 2t): t \in \mathbb{R}\}$. Both are convex (hence midpoint convex) and intersect at (0,0). Pick $p = (2,0) \in A_1$ and $q = (-2, -4) \in A_2$. Then $p + q/2 = (0, -2) \notin A_1 \cup A_2$. (See Fig. 2.)

Theorem 3.4 Union of a chain

The intersection of any family of midpoint-convex sets is also midpoint-convex.

Proof.
Let $\{A_i: i \in I\}$ be a family of midpoint convex sets in a linear space L, and define

$$A = \bigcap_{i \in I} A_i.$$

For any $x, y \in A$, we have $x, y \in A_i$ for each i.
Since each A_i is midpoint convex, $x + y/2 \in A_i$ for all i.
Hence, $x + y/2 \in \bigcap_{i \in I} A_i = A$.
Therefore, A is midpoint convex.

Theorem 3.5. (Image and Inverse Image under Linear Transformation)

Let $T: X \to Y$ be a linear transformation between linear spaces.
Then:

1. The image of a midpoint convex set in X is midpoint convex in Y; and
2. The inverse image of a midpoint convex set in Y is midpoint convex in X.

Proof.
Let $A \subseteq X$ be midpoint convex.
For $z_1, z_2 \in T(A)$, there exist $x_1, x_2 \in A$ such that $z_1 = T(x_1)$ and $z_2 = T(x_2)$.
Then

$$\frac{z_1 + z_2}{2} = T\left(\frac{x_1 + x_2}{2}\right).$$

Since A is midpoint convex, $x_1 + x_2/2 \in A$. Thus $z_1 + z_2/2 \in T(A)$.
Hence, the image is mid-point convex.

Now let $B \subseteq Y$ be midpoint convex and consider $T^{-1}(B)$.
For $x_1, x_2 \in T^{-1}(B)$, we have $T(x_1), T(x_2) \in B$.
As B is midpoint convex,

$$\frac{T(x_1) + T(x_2)}{2} = T\left(\frac{x_1 + x_2}{2}\right) \in B.$$

Therefore, $x_1 + x_2/2 \in T^{-1}(B)$, proving that $T^{-1}(B)$ is midpoint convex.

Theorem 3.6 (Translation of a set).

If A is a subset of a linear space L and $x \in L$, the set

$$x + A = \{\, x + a : a \in A \,\}$$

is called the **translate** of A by x [4].

Theorem 3.7.

If A is a midpoint convex subset of L, then every translate $x + A$ is also midpoint convex.

Proof.
Let $y_1, y_2 \in x + A$.
Then $y_1 = x + a_1$ and $y_2 = x + a_2$ for some $a_1, a_2 \in A$.
Now,

$$\frac{y_1 + y_2}{2} = x + \frac{a_1 + a_2}{2}.$$

Since A is midpoint convex, $a_1 + a_2/2 \in A$.
Hence, $y_1 + y_2/2 \in x + A$.
Therefore, $x + A$ is midpoint convex.

4. From Midpoint Convexity to Convexity

Midpoint convexity gives all **dyadic combinations**.

$$m/2^n = 1/2\,(1/2\,(\cdots)\cdots), \underbrace{}_{\text{n midpoints}}$$

so $\alpha x + (1-\alpha)y \in A$ for every dyadic $\alpha \in [0,1]$. Approximating general α by dyadics then yields convexity under mild closure or regularity.

Theorem 4.1 (Closed midpoint convex ⇒ convex) [1,3].

If $A \subseteq L$ is midpoint convex and **closed** (in a TVS), then A is convex. *Proof sketch*. Let α_n be dyadics with $\alpha_n \to \alpha$. Since $\alpha_n x + (1-\alpha_n)y \in A$ and A is closed, the limit $\alpha x + (1-\alpha)y \in A$.

Theorem 4.2 (Local boundedness or measurability) [5,6].

If $A \subseteq \mathbb{R}^n$ is midpoint convex and either (i) **locally bounded** at one point or (ii) **Lebesgue-measurable** with nonempty interior, then A is convex.

Idea. Apply Jensen-type upgrading: midpoint convexity + mild regularity ⇒ full convexity (classical for functions; adapt via indicator sets) [5,6].

(Figure 3 illustrates dyadic constructions converging to a general α.)

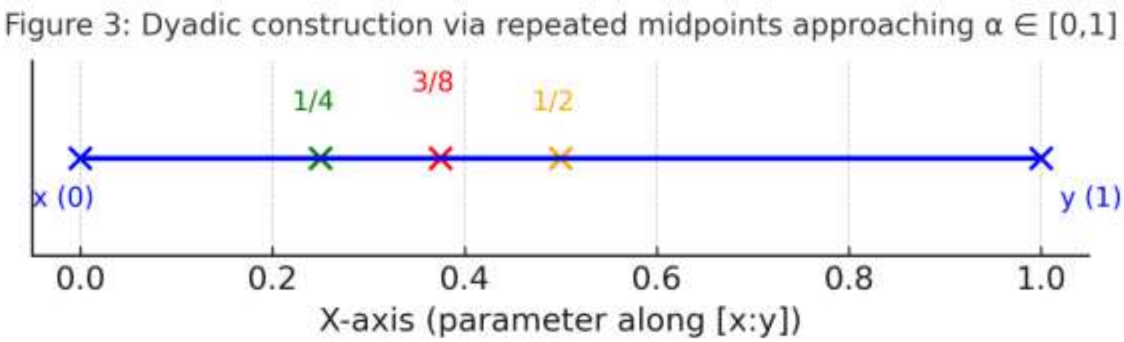

Figure 3. Dyadic construction: repeated midpoints generate $\alpha = m/2^n$ and approximate any $\alpha \in [0,1]$; closedness then yields full convexity.

5. Examples and Counterexamples

Example 5.1 (Convex ⇒ midpoint convex). Every affine subspace or half-space is convex; hence, it is midpoint convex [1,3].

Example 5.2 (Midpoint convex but not convex).
Let $f: \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function (Hamel basis construction) [6]. Its **epigraph** $E = \{(x,t): t \geq f(x)\}$ is midpoint convex (Jensen additivity) yet **not** convex due to lack of regularity of f.

Counterexample 5.3 (Union failure):
Union of two distinct lines through the origin in $\mathbb{R}^2$ fails midpoint convexity (Fig. 2).

6. Discussion and Concluding Remarks

6.1 Conceptual significance

Midpoint convexity captures a **"first-order" convex behaviour-stability** under averaging two points without committing to all convex combinations. Thus, it is well-suited to **iterative methods** that proceed by halving or averaging steps (e.g. bisection-like feasibility updates, Krasnosel' skiĭ–Mann iterations, and other projection schemes where the midpoint is a natural choice). In such contexts, midpoint-convex feasible regions allow the algorithm to remain feasible under midpoint updates, even when the full convexity of the feasible set is unknown or too strong.

6.2 When midpoint convexity suffices-and when it does not

If the workflow **only** uses midpoints (e.g. repeated halving) and the data/space confer mild regularity, midpoint convexity may be **functionally adequate**.

- Intersections preserve midpoint convexity (Theorem 3.2), so multi-constraint settings remain tractable in this case.
- Linear images/preimages preserve midpoint convexity (Theorem 3.4), so switching coordinates or enforcing linear constraints does not break the structure of the problem.

- Translations preserve midpoint convexity (Theorem 3.5), which is valuable for sensitivity analysis or recentering.

However, midpoint convexity **fails** to guarantee closure under many standard convex operations that crucially use **non-dyadic** coefficients, and **unions** jeopardise the property unless nested (Theorem 3.7). Therefore, true convexity is indispensable in applications that require arbitrary convex combinations.

6.3 Links to Jensen convexity and dyadic limits

The midpoints-to-dyadics passage parallels the core idea behind **Jensen's convex** functions (midpoint convex functions). Classical theorems assert: Jensen convexity + **measurability/local boundedness** ⇒ convexity [5,6]. This is mirrored here: midpoint convex sets + **closedness** (or the set analogue of "nice" regularity) ⇒ convex sets. Consequently, midpoint convexity can be seen as a **pre-convex** property that becomes convex "in the limit" when a minimal analytic structure is present.

6.4 Geometric and functional contexts

- **Convex geometry**: Midpoint convex bodies share some geometric intuitions with convex bodies but not the full Brunn-Minkowski apparatus [8]. However, averaging arguments and barycentric ideas (e.g. in uniformly convex spaces [7]) resonate with midpoint logic.
- **Optimisation**: In feasibility-seeking, where constraints are only known to be midpoint convex (e.g. empirical constraints closed under pairwise averaging), one can design averaging-based algorithms that maintain feasibility and then attempt to show convexity via regularity checks (closedness, measurability).
- **Variational analysis**: The epigraph perspective connects midpoint convexity to Jensen-type structures and sublevel sets of midpoint convex functions [5,6].

6.5 Correcting the union misconception

The counterexample in $\mathbb{R}^2$ (Figure 2) shows that even very "nice" midpoint convex sets (lines) can have a union that is **not** midpoint convex. Therefore, arguments that rely on "gluing" feasible regions via union should seek nested constructions or rely on intersections (which are safe).

6.6 Limitations and open problems

Several natural research problems have emerged.

1. **Midpoint Carathéodory**

   For convex sets in $\mathbb{R}^d$, Carathéodory's theorem bounds the number of points needed to represent a convex combination. What is the **sharp dyadic/halving analogue** for closed midpoint convex sets (or those that satisfy upgrading conditions)?

2. **Quantitative upgrading.**

   Given a closed midpoint convex set $A \subset \mathbb{R}^n$, what is an **effective rate** by which dyadic approximations (via midpoints) fill all convex combinations? Can one bound the number of halvings needed to ε-approximate $\alpha \in [0,1]$?

3. **Stability under nonlinear maps**

   While linear images and preimages preserve midpoint convexity, general **nonlinear** maps do not preserve it. Which classes (e.g. affine, monotone operators, proximal mappings) preserve the midpoint convexity of images or preimages?

4. **Separation phenomena**

   Separation theorems are central to convex analysis [1,3]. Are there **weak separation** results for midpoint-convex sets under additional regularity? What role do supporting functionals play in this process?

5. **Measure-theoretic refinements**

   Extending Jensen-type regularity upgrades for sets, can we classify minimal measure-theoretic assumptions ensuring midpoint convex ⇒ convex in $\mathbb{R}^n$?

Addressing these issues would better place midpoint convexity within the landscape of convex analysis and could suggest new algorithmic heuristics.

Midpoint convexity is a compelling and surprisingly robust weakening of convexity that preserves many useful operations, intersections**, linear images/preimages, translations, and certain Minkowski combinations**. This supports a constructive **dyadic** approach to building general convex combinations through repeated halving. While midpoint convex sets may fail to be convex in general, **closedness** (in a TVS) or **mild regularity** (measurability/local boundedness) often **upgrades** the midpoint convexity to full convexity. This bridges a practical gap: in settings where only midpoint stability is known or easily verified, convexity can still be obtained by checking the light regularity conditions. The corrected understanding of **unions** (unsafe unless nested) prevents common pitfalls. We anticipate that further studies, particularly quantitative dyadic approximations, midpoint analogues of classical convex theorems, and stability under structured nonlinear maps, will enhance both the theoretical foundations and algorithmic applications of this elegant concept.

7. Applications and Outlook

Midpoint convexity appears in averaging algorithms, projection and splitting methods that use repeated halving, and relaxations where dyadic stability is sufficient (e.g. feasibility heuristics). The transformation properties (Theorems 3.5-3.7) allow midpoint convexity to propagate through linear mappings and translations, while Theorems 4.1-4.2 provide practical criteria to recover full convexity in TVSs or $\mathbb{R}^n$. Future directions include midpoint analogues of Carathéodory's theorem, stability under nonlinear images, and links to Jensen's convex functionals in variational analysis [5,6].

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