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Fuzzy Lucky Labeling And Proper Fuzzy Lucky Labeling Of Book And Friendship Graph

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ABSTRACT

This paper investigates fuzzy lucky labeling for book graphs and friendship graphs. We prove that two fuzzy values suffice for multi-page book graphs, while three are required for proper fuzzy lucky labeling of friendship graphs. These constructive results establish the minimal labeling numbers and contribute to the theory of fuzzy graphs, with potential applications in network modeling.

Keywords: Fuzzy Lucky Labeling, Proper Fuzzy Lucky Labeling, Fuzzy lucky number, Proper Fuzzy lucky number, Book graph, Friendship graph.

INTRODUCTION

This research journey explored the fascinating puzzle of how to assign fuzzy labels—values between 0 and 1 representing degrees of membership—to different types of networks, or graphs, following a special set of rules. We focused on two classic and visually intuitive graph families. The first, book graphs, are like a stack of pages (triangles or squares) all joined along a common spine. For these, we discovered a beautifully simple and efficient solution: for any book with more than one page, you only ever need two different labels. By arranging them in a careful, alternating pattern, we can guarantee the "lucky" condition is met everywhere, proving that sometimes a complex-looking structure has a surprisingly straightforward solution.

The second family, friendship graphs, are charming networks of interlocking triangles all connected through a single central point. Here, the story became more nuanced. We found that the basic lucky labeling rule could also be satisfied with just two labels. However, when we added a stricter rule—that directly connected points must also have *different* labels (a proper coloring)—the graph forced our hand. Because of the fundamental nature of triangles, a third label became absolutely essential. This highlights a crucial insight: the minimal number of labels isn't always about the luckiness alone; it can also be dictated by the very shape and connectivity of the network itself.

In the end, this work is more than just a theoretical exercise. By providing these optimal labeling schemes, we contribute key building blocks to the field of fuzzy graph theory. These concepts have exciting potential for application in designing robust networks, managing uncertain data, and modeling complex systems where relationships are not just black and white, but exist in shades of gray.

Definition: Fuzzy lucky labeling

Let G = (V,E) be a graph. A fuzzy lucky labeling of G is a function

$$f : V(G) \to (0,1]$$

that assigns a fuzzy value to each vaertex.For each vertex u ∈ $v(G)$, define

$$s_f(u) = \sum_{uv \in E(G)} f(v).$$

If $s_f(u) \neq s_f(v)$ for every edge $uv \in E(G)$, then f is called a fuzzy lucky labeling of G.

Definition: Proper fuzzy lucky labeling

Let G = (V,E) be a graph with a fuzzy lucky labeling f. A proper fuzzy lucky labeling of G is a fuzzy lucky labeling f such that

$$f(u) \neq f(v) \text{ for every edge } uv \in E(G).$$

In other words, in addition to the condition $s_f(u) \neq s_f(v)$, adjacent vertices must receive distinct fuzzy labels.

Definition: Lucky Numbers

Let G = (V,E) be a graph. A fuzzy lucky labeling of G is a function

$$f : V(G) \to [0,1]$$

such that for every edge uv∈ $E(G)$ we have $s_f(u) \neq s_f(v)$, where $s_f(u) = \sum_{uv \in E(G)} f(v)$.

The fuzzy lucky number of G, denoted by $\eta_f(G)$, is defined as

$$\eta_f(G) = \min\{|f(V(G))| : f \text{ is a fuzzy lucky labeling of } G\},$$

That is,the minimum cardinality of the set of fuzzy labels require to obtain a fuzzy lucky labeling of G.

Book Graphs: Triangular and Rectangular

Definition: Triangular Book Graph

For t>1, the triangular book B_3^t is obtained by taking t copies of the triangle C_3 and identifying the same edge(the spine) in all copies. A model is

$$V(B_3^t) = \{x, y\} \cup \{w_i : 1 \leq i \leq t\},$$

$$E(B_3^t) = \{xy\} \cup \{xw_i, yw_i : 1 \leq i \leq t\}.$$

Hence $|V| = 2 + t$ and $|E|$=1+2t.

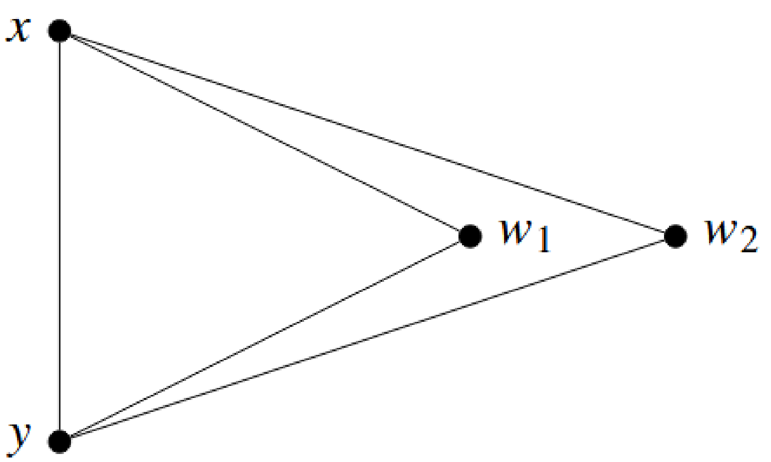

Figure1: Triangular book B_3^2.

Definition: Rectangular Book Graph

For t≥1, the rectangular book B_4^t is obtained by taking t copies of the cycle C_4 and identifying the same edge (the spine) in all copies. A model is $V(B_4^t) = \{x, y\} \cup \{p_{i,} q_i : 1 \leq i \leq t\}$,

$$E(B_4^t) = \{xy\} \cup \{xp_i, : p_i q_i, q_i y 1 \leq i \leq t\}.$$

Hence $|V| = 2 + 2t$ and $|E|$=1+3t.

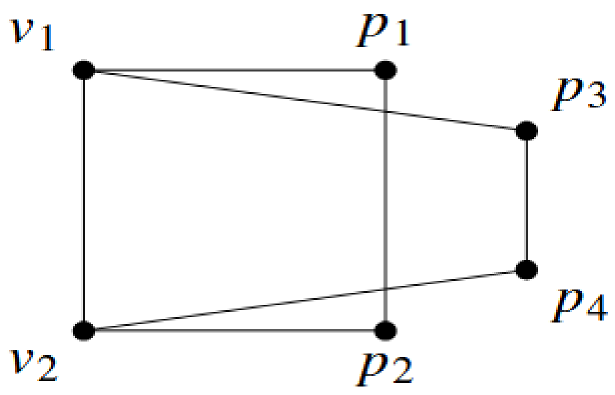

Figure2: Rectangular (quadrilateral) book B_4^2.

Theorem 1:

For integers n≥ 3 and t≥ 1, the book graph B_n^t admits a fuzzy lucky labeling. Moreover, if t≥ 2 there exist a 2-valued fuzzy lucky labeling(hence $\eta_f(B_n^t) = 2$). For the single –page case

$$B_n^1 = C_n, \eta_f(C_n) = 2$$

When n is even and $\eta_f(C_n) = 3$ when n is odd.

Proof. Model each page by an (x,y)-path

$$x = w_{i,o}, w_{i,1}, \dots, w_{i,n-2}, w_{i,n-1} = y$$

And let xy be the common spine edge.

Two-label construction for t≥ 2 . Choose distinct fuzzy numbers $\mu_1, \mu_2 \in (0,1]$ such that (t+1) $\mu_2 \neq 2\mu_1$ and (t+1) $\mu_1 \neq 2\mu_2$

Define

$$f(x) = \mu_1,\ f(y) = \mu_2,$$

$$f(w_{i,j}) = \begin{cases} \mu_2, j\ odd \\ \mu_1, j\ even \end{cases}, \qquad (1 \le i \le t, 1 \le j \le n-2).$$

For interior vertices $w_{i,}$ with $1 \le j \le n-3$ we have

$$s_f(w_{i,j}) = \begin{cases} 2\mu_{1,}, j\ odd \\ 2\mu_2, j\ even \end{cases}$$

So $s_f(w_{i,j}) \neq s_f(w_{i,j+1})$ along each internal edge.

At the page ends,

$$s_f(w_{i,1}) = 2\mu_1, \qquad s_f(w_{i,n-2}) = \begin{cases} 2\mu_2,\ \ n\ even, \\ \mu_1 + \mu_2,\ n\ odd, \end{cases}$$

At the spine,

$$s_f(x) = (t+1)\mu_2, \qquad s_f(x) = \begin{cases} (t+1)\mu_2,\ \ n\ even, \\ \ \mu_1 + \mu_2,\ n\ odd, \end{cases}$$

Hence, on every edge the neighbor-sums differ:

$$xy = s_f(x) \neq s_f(y)\ \ (\mu_1 \neq \mu_2)\ ,$$

$$xw_{i,1}: (t+1)\mu_2 \neq 2\mu_1 \ \ \text{by choice},$$

$$yw_{i,n-2} := \begin{cases} (t+1)\mu_1 \neq 2\mu_2, & n\ even\ (by\ choice), \\ \mu_1 + t\mu_2 \neq \mu_1 + \mu_2, & n\ odd\ since\ t \geq 2, \end{cases}$$

$$w_{i,j}w_{i,j+1}: 2\mu_{1,} \neq 2\mu_2.$$

Thus f is a fuzzy lucky labeling using two fuzzy values, proving $\eta_f(B_n^t) \leq 2$ for t $\geq$ 2.

A constant labeling (1 value) fails: if n$\geq$ 4 there is an internal edge whose ends both have degree 2; if n=3 the spine ends x and y are adjacent with equal degree t+1, giving equal neighbor- sums. Hence $\eta_f(B_n^t) \geq 2$, so $\eta_f(B_n^t) = 2$ for t $\geq$ 2.

Single page t=1.Here $B_n^1 = C_n$. A 2-label alternating assignment works when n is even; when n is odd, three labels are needed. This proves existence of a fuzzy lucky labeling for all n $\geq$ 3 and t $\geq$ 1.

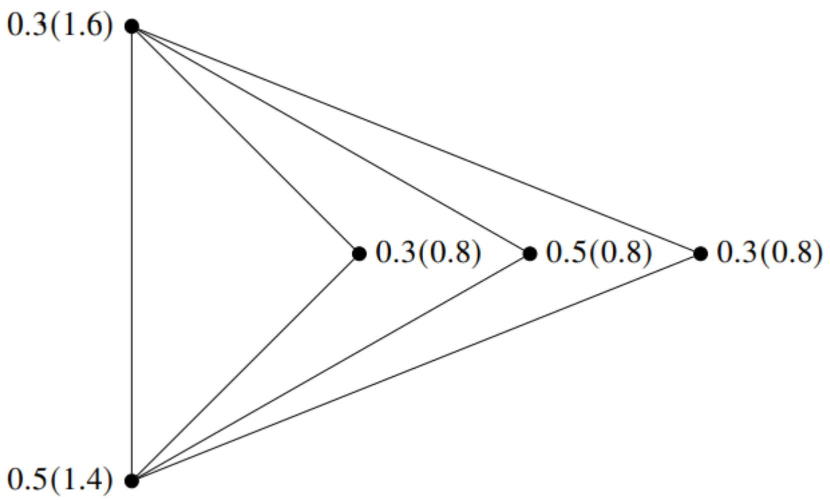

Figure3: FL Labeled Triangular book B_3^3.

Theorem 2:

For every t$\geq$ 1, the rectangular (quadrilateral) book B_4^t admits a fuzzy lucky labeling with two fuzzy values. Hence $\eta_f(B_4^t) = 2$.

Proof: Model each of the t pages as a path xp_iq_iy that, together with the spine edge xy, forms a 4-cycle (1$\leq i \leq t$). Thus

$$V = \{x, y\} \cup \{p_i, q_i: 1 \leq i \leq t\},$$

$$E = \{xy\} \cup \{xp_i, p_iq_i, q_iy: 1 \leq i \leq t\}.$$

If all vertices receives the same fuzzy value μ, then $s_f(p_i) = \mu + \mu = 2\mu = s_f(q_i)$, so the edge p_iq_i violates the lucky condition. Hence $\eta_f(B_4^t) \geq 2$.

(Two label construction) Choose distinct fuzzy numbers $\mu_1, \mu_2 \in (0,1]$ with

$$(t+1)\mu_2 \neq 2\mu_1 \quad \text{and } (t+1)\mu_1 \neq 2\mu_2$$

Define

$f(x) = \mu_1,\ f(y) = \mu_2,\ \ f(p_i) = \mu_2,\ f(q_i) = \mu_1,\ (1 \leq i \leq t).$

Neighbor- sums are

$$s_f(x) = f(x) + \sum_{i=1}^{t} f(p_i) = (t+1)\mu_2,$$

$$s_f(y) = f(x) + \sum_{i=1}^{t} f(q_i) = (t+1)\mu_1,$$

$$s_f(p_i) = f(x) + f(q_i) = 2\mu_1,$$

$$s_f(q_i) = f(p_i) + f(y) = 2\mu_{2.}$$

Check each edge:

$$xy: (t+1)\mu_2 \neq (t+1)\mu_1 \quad (\mu_1 \neq \mu_2),$$

$$xp_i: (t+1)\mu_2 \neq 2\mu_1 \text{ (by choice)},$$

$$p_iq_i: 2\mu_1 \neq 2\mu_2 \quad (\mu_1 \neq \mu_2),$$

$$q_iy: 2\mu_2 \neq (t+1)\mu_1 \text{ (by choice)}.$$

Thus f is a fuzzy lucky labeling using two fuzzy values, proving $\eta_f(B_4^t) \leq 2$. Thogether with the lower bound ,we obtain $\eta_f(B_4^t) = 2$.

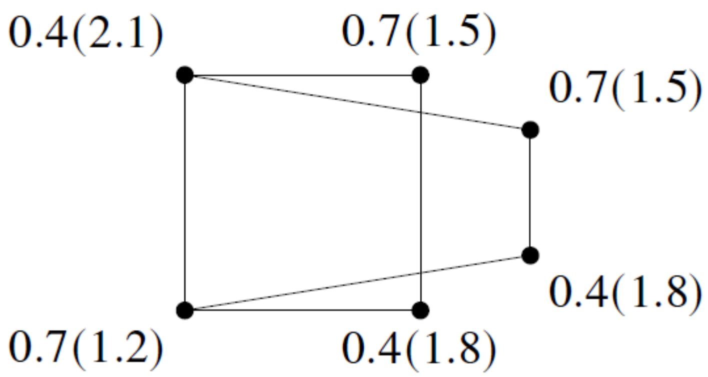

Figure4: FL Labeled Rectangular (quadrilateral) book B_4^2.

FRIENDSHIP GRAPH

Definition: Friendship Graph

For an integer ≥ 1 , the friendship graph $F_n$is the graph formed by taking n copies of the triangle $C_3$and identifying one common vertex. Equivalently, with a central vertex v and distinct pairs (a_i, b_i) for $i = 1, \dots, n$,

$V(F_n) = \{v\} \cup \{a_i, b_i: 1 \leq i \leq n\}, \quad E(F_n) = \{va_i, vb_i, a_ib_i: 1 \leq i \leq n\}.$

Hence $|V(F_n)| = 2n + 1, |E(F_n)| = 3n, \deg(v) = 2n, and \deg(a_i) = \deg(b_i) = 2.$

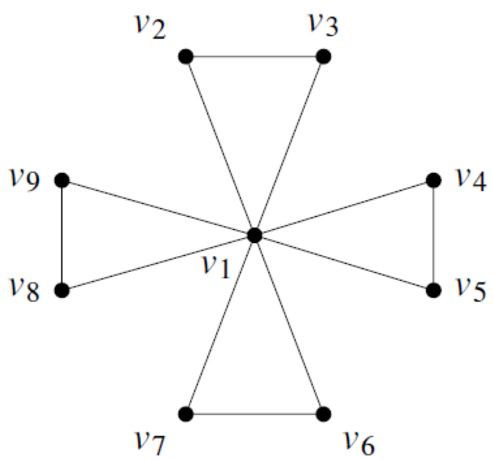

Figure 5: Friendship graph F_4.

Theorem 3:

For $n \geq 2$, the friendship graph F_n (the union of n triangles with a common vertex) admits a 2-valued fuzzy lucky labeling: hence

$\eta_f(F_n) = 2.$

For the single page $F_1 = c_3$, we have $\eta_f(F_1) = 3.$

Proof. Let the common vertex be v, and for each triangle (page) $i \in \{1, \dots, n\}$ let its other two vertices be a_i and b_i. Then

$V(F_n) = \{v\} \cup \{a_i, b_i: 1 \leq i \leq n\}, \quad E(F_n) = \{va_i, vb_i, a_ib_i: 1 \leq i \leq n\}.$

Choose fuzzy numbers $\mu_1, \mu_2 \in (0,1]$ with

$$\mu_1 \neq \mu_2 \text{ and } n(\mu_1 + \mu_2) \neq 2\mu_1$$

Define a labeling

$f(v) = \mu_1, \quad f(a_i) = \mu_1, \quad f(b_i) = \mu_2 \quad (1 \leq i \leq n).$

Compute the induced neighbor-sums $s_f(x) = \sum_{xy \epsilon E(F_n)} f(y)$:

$s_f(v) = \sum_{i=1}^{n}\big(f(a_i) + f(b_i)\big) = n(\mu_1 + \mu_2)$, $\quad s_f(a_i) = f(v) + f(b_i) = \mu_1 + \mu_2$,

$s_f(b_i) = f(v) + f(a_i) = \mu_1 + \mu_2$,

Check each edge:

$$a_i b_i: s_f(a_i) = \mu_1 + \mu_2 \neq 2\mu_1 = s_f(b_i) \quad (\mu_1 \neq \mu_2),$$

$$v a_i: s_f(v) = n(\mu_1 + \mu_2) \neq \mu_1 + \mu_2 = s_f(a_i) \qquad (n \geq 2),$$

$v b_i: s_f(v) = n(\mu_1 + \mu_2) \neq 2\mu_1 = s_f(b_i)$ (by choice).

Thus f is a fuzzy lucky labeling using two fuzzy values, so $\eta_f(F_n) \leq 2$. A one-value labeling $f \equiv \mu$ fails because on each edge $a_i b_i$ we obtain $s_f(a_i) = s_f(b_i) = 2\mu$, violating the lucky condition; hence $\eta_f(F_n) \geq 2$. Therefore $\eta_f(F_n) = 2$ for n≥ 2. For n=1, $F_1 = C_3$ and two values are insufficient while three suffice, so $\eta_f(F_1) = 3$.

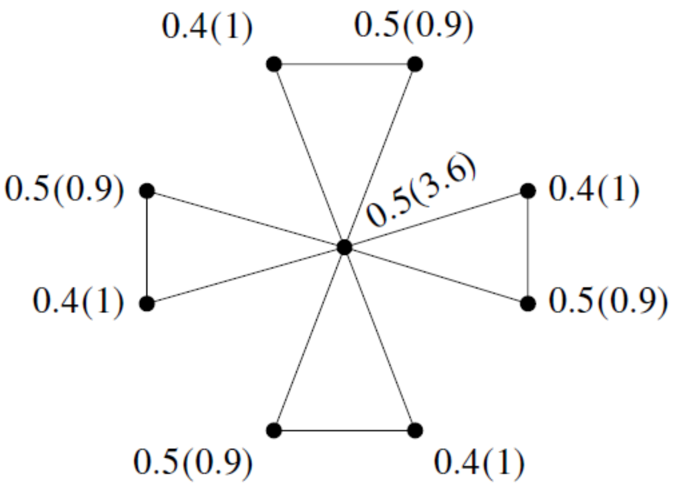

Figure 6: FL Labeled Friendship graph F_4.

Hence f is a fuzzy proper lucky labeling, so $\eta_f^*(F_n) \leq 3$.

For the lower bound, each page contains a triangle $v a_i b_i$, so $\chi(F_n) = 3$. Any fuzzy proper lucky labelling must be a proper coloring, hence uses at least $\chi(F_n)$ distinct fuzzy values. Therefore $\eta_f^*(F_n) \geq 3$, and we conclude $\eta_f^*(F_n) = 3$.

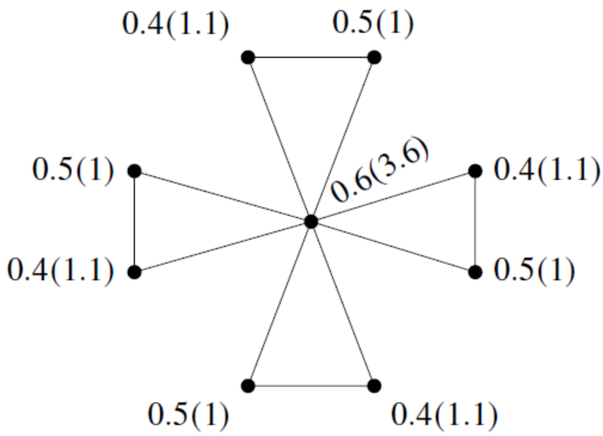

Figure 7: FPL Labeled Friendship Graph F_4.

CONCLUSION

This paper established the minimal number of fuzzy labels required for lucky and proper lucky labelings on book graphs and friendship graphs. It was shown that for multi-page books, only two labels are needed, while the proper labeling of friendship graphs requires three. These efficient constructions contribute to fuzzy graph theory and provide a foundation for applying these concepts to network models and other complex structures. Future work could extend these findings to other graph families and explore their practical applications.

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