IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Solution Of The Duffing EquationBy The Power Series Method

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Abstract:

The generic version of the Duffing equation is examined in this article. This problem can be solved with beginning conditions using an adaptation of the power series approach. The power series method's convergence theorem is demonstrated. The Duffing equation is a nonlinear second-order differential equation that models the behavior of a driven, damped oscillator with cubic nonlinearity. In this study, we apply the power series method to obtain an approximate solution to the Duffing equation. The power series expansion expresses the displacement x(t) as an infinite sum of powers of time, which is substituted into the equation to generate a system of recursive equations for the coefficients of the series. By solving these equations iteratively, we determine the approximate solution for small driving force, weak nonlinearity, or for short times. The method provides an effective way to analyze the system's behavior under small perturbations and serves as a foundation for exploring more complex nonlinear oscillatory systems. The approach is validated through the calculation of the first few terms in the series expansion, providing insight into the dynamics of the oscillator for various values of the system parameters such as damping, linear stiffness, and nonlinear stiffness.

Keywords: Duffing Equation, Differential Equation ,power series

Introduction:

Georg Duffing (1918) conducted extensive research on mechanical systems with nonlinearities, which marked the beginning of the 20th century's serious study on forced nonlinear oscillators.One excellent nonlinear differential equation that appears in many physical, engineering, and even biological problems is the Duffing equation. In 1918, German electrical engineer Duffing presented the Duffing equation model.The goal of this article is to provide a different method for solving the Duffing equation. The Duffing Equation's initial value difficulty was solved by applying the power series approach, yielding infinite series as a result. White and Schovanec employed the Taylor series expansiontechnique. A proposed analytical method for the analysis of periodic motion problems is basedon the power series method. Qaisi has solved the undamped and unforced Duffing equation using the power series method.The harmonic balance method, variation iteration method, homotopy perturbation method, parameter-expanding method, exp-function method, differential transform method, and optimal scale polynomial interpolation method are just a few of the computational techniques that have been developed to solve the oscillatory problems of nonlinear oscillators. Khuri introduced the Laplace decomposition methods. Chen has solved the Duffing problem using the target function approach. Yusufoglu used the Laplace decomposition approach to solve the Duffing equation quantitatively. Kumar has created a potent technique that uses the Laplace transform to solve the time-fractional Fokker-Planck equation that arises in solid state physics, circuit theory, gas dynamics equation that arises in shock fronts, and telegraph equation. Chen has solved the Duffing problem using the target function approach. Yusufoglu used the Laplace decomposition approach to solve the Duffing equation quantitatively. Kumar has created a potent technique that uses the Laplace transform to solve the time-fractional Fokker-Planck equation that arises in solid state physics, circuit theory, gas dynamics equationthat arises in shock fronts, and telegraph equation.

It is now our turn to attempt to solve the Duffing equation using the power series method (PSM). PSM is a traditional approach to solving ordinary differential equations that is similar to the Taylor series method except that it does not require the expansion coefficients to be derived through a complex differential process. Because it is required to develop the proper algebraic connection in order to create the expansion coefficients case by case, the PSM is not offered as a general-purpose procedure.

Mathematical modeling

A non-linear second-order differential equation with the following initial conditions will be defined in this section

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos\omega t \quad , x(0) = p_0 \,;\, \dot{x}(0) = p_1$$

where the values of p_0 and p_1 are set, δ is control the amount of damping, α is control the linear stiffness, β is control the amount of non-linearity in the restoring force; if $\beta = 0$ the Duffing equation is driven simple harmonic oscillator; γ is the amplitude of the periodic drivingforce; if $\gamma = 0$ the system is without a driving force; ω is angular frequency of the periodic driving force.

The power series of solutions for non-linear equations converge on a small intervals. Therefore the initial time must be transformed in the calculations. Let's write the problem in the general form as:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t + t_0) \quad , x(0) = p_0 \,;\, \dot{x}(0) = p_1 \qquad (1)$$

Where t_0 is the initial time, t is measured from the moment t_0,

$$\cos(\omega t + t_0) = \sum_{i=0}^{\infty} \frac{\left(\cos(\omega t + t_0)\right)^{(i)}\Big|_{t=0}}{i!} t^i$$

Let

$$u_0 = 1, u_1 = \frac{\omega u_1}{1}, u_2 = \frac{\omega u_1}{2}, \dots, u_i = \frac{\omega u_{i-1}}{i}, \dots$$

Hence

$$\cos(\omega t + t_0) = \sum_{i=0}^{\infty} u_i \cos\left(\omega t_0 + \frac{\pi i}{2}\right) t^i \qquad (2)$$

A modified power series method

The power series method is a standard and effective tool for solving the Duffing equation .It gives the solution x(t) in the form of a power series

$$x(t) = \sum_{i=0}^{\infty} p_i t^i \tag{3}$$

Then $\dot{x}(t) = \sum_{i=0}^{\infty}(i+1)p_{i+1}t^i = \sum_{i=0}^{\infty} q_i t^i$ (4)

Where

$$q_i = (i+1)\, p_{i+1}$$

Note that here

$$q_{i+1} = (i+2)\, p_{i+2}$$

Then

$$\ddot{x}(t) = \left(\sum_{i=0}^{\infty} q_i t^i\right)' = \sum_{i=0}^{\infty}(i+1)q_{i+1}t^i = \sum_{i=0}^{\infty}(i+1)(i+2)p_{i+2}t^i \tag{5}$$

Now taking square and cube of (3) we get the following results:

$$x^2 = x.x = \sum_{i=0}^{\infty}\left(\sum_{i=0}^{i} p_k p_{i-k}\right)t^i = \sum_{i=0}^{\infty} r_i t^i$$

$$x^3 = x^2.x = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} r_j p_{i-j}\right)t^i = \sum_{i=0}^{\infty}\left[\sum_{j=0}^{i} p_{i-j} \sum_{k=0}^{k} p_k p_{j-k}\right]t^i \tag{6}$$

Now using (2)-(6) in equation (1), we get the following result

$$p_{i+2} = -\frac{\delta_{p_{i+1}}}{i+2} - \frac{\beta \sum_{j=0}^{i} p_{i-j} \sum_{k=0}^{j} p_k p_{j-k}}{(i+1)(i+2)} + \frac{\gamma u_i \cos\left(\omega t_0 + \frac{\pi i}{2}\right)}{(i+1)(i+2)} \tag{7}$$

For $i = 0$ p_2 is expressed through p_0 and p_1.

Note that ! grows faster than ω^i, i.e.

$$\lim_{i \to \infty} \frac{\omega^i}{i!} = 0$$

Then there exist such the value $i = i^*$, when

$$\frac{\omega^i}{i!} < 1$$

For any $i > i^*$.

We denote by:

$$N_\omega = \max_{i=0,i^*} \frac{\omega^i}{i!} \geq 1; \frac{\omega^i}{0!} = 1$$

$$\left|cos\left(\omega t_0+\frac{\pi i}{2}\right)\right|\leq 1, |u_i|\leq N_\omega$$

Estimating the region of convergence of the power series is important when the integration step is selected.

Let Δt is the integration step,

$$\Delta t\in\left(-\frac{1}{h};\frac{1}{h}\right)$$

h is some number, which we also define:

$$h_1 \max\{|p_0|,|p_1|,1\}\geq 1; h_2=\delta+\beta\left(3{h_1}^2+3h_1+1\right)+\gamma N_\omega+1\geq 1; h=h_1h_2\geq 1$$

Theorem : The following inequalities are hold

$$|p_v|\leq h^v \qquad (8)$$

Where v is any natural number.

Proof. We prove by the method of mathematical induction.

$|p_1|\leq h^1$ is obviously.

We assume that (8) is valid for $i=m+1, m$ is a natural number. Then it is also valid for any $n=\overline{1,\imath}$ that is

$$|p_n|\leq h^n$$

Note

$$\sum_{j=0}^{i} p_{i-j}\sum_{k=0}^{j} p_kp_{j-k}=p_ip_0^2+p_0\left(2p_0p_i+\sum_{k=1}^{i-1}p_kp_{i-k}\right)+\sum_{j=1}^{i-1}p_{i-j}\sum_{k=0}^{j}p_kp_{j-k}$$

$$=p_ip_0^2+p_0\left(2p_0p_i+\sum_{k=1}^{i-1}p_kp_{i-k}\right)+\sum_{j=1}^{i-1}p_{i-j}\left(2p_0p_j+\sum_{k=1}^{j-1}p_kp_{j-k}\right)$$

$$h^j+\sum_{k=1}^{j-1}h^kh^{j-k}\Bigg)$$

$$=h_1^2h^i+h_1\left(2h_1h^i+h^i(i-1)\right)+2h_1\sum_{j=1}^{i-1}h^i+\sum_{j=1}^{i-1}\sum_{k=1}^{j-1}h^i$$

Then from (7)

$$|p_{m+2}|\leq\frac{\delta|p_{m+1}|}{m+2}+\beta(3h_1^2+3h_1+1)h^m+\frac{\gamma|u_m|\left|\cos\left(\omega t_0+\frac{\pi m}{2}\right)\right|}{(m+1)(m+2)}\leq\delta h^{m+1}+$$

$$+\beta(3h_1^2+3h_1+1)h^{m+1}+\gamma N_\omega h^{m+1}\leq(\delta+\beta(3h_1^2+3h_1+1)+\gamma N_\omega+1)h^{m+1}=$$

$$=h_2h^{m+1}\leq h_1h_2h^{m+1}=h^{m+2}$$

$$h_1^2+3h_1+1)h^i.$$

Hence proved that

$$|p_v|\leq h^v$$

Where v is any natural number.

CONCLUSION :

we have demonstrated the application of the power series method to solve the Duffing equation, a widely studied nonlinear differential equation describing a driven, damped oscillator with both linear and cubic restoring forces. By assuming a power series expansion for the displacement x(t)x(t)x(t), we derived a recursive system of equations for the coefficients, which were solved iteratively. This method provides an efficient way to obtain approximate solutions for small forcing amplitudes, weak nonlinearities, or for short time intervals where other solution techniques may be challenging. The power series method proves particularly useful for analyzing the behavior of the Duffing oscillator under conditions where exact solutions are not available. It offers insights into the system's dynamics, including periodic and quasi-periodic motion, resonance phenomena, and the influence of damping and nonlinearity on the system's response. However, the method's applicability is limited to cases where the perturbation parameters (such as the driving force amplitude or the nonlinearity coefficient) are small enough to ensure convergence of the series. For larger values of these parameters, the series may converge slowly or diverge, necessitating the use of numerical methods or alternative perturbation techniques for more accurate results. The power series method is a valuable tool for approximating the solution of the Duffing equation in the context of small perturbations. It lays the groundwork for further exploration of nonlinear oscillatory systems, offering a simple yet effective approach for obtaining analytical approximations in cases where full exact solutions are difficult to derive.

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