IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

Study Of Effectiveness Of Sperner's Lemma Using Triangulations

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Abstract: In this paper, we first introduce the concept of simplex,simplicial subdivision,Sperner lemma etc..Further we have shown that Kakutani theorem can be proved by using sperners lemma .Sperners lemma is also a powerful tool in proving Gale-nikaido-debreu lemma. Additionally,it is shown that Kakutani theorem is a corollary of Gale-nikaido-debreu lemma.

Key words:- Sperner lemma, Simplex, Triangulation, Fixed Point Theorem, Gale-Nikaido-Debreu Lemma.

I. Introduction

Fixed point theorems have a lot of importance in mathematical sciences. These theorems have been quite beneficial in better aproximation of fixed points.we prove Kakutani theorem as well as different versions of Gale-Nikaido-Debreu (GND) lemma by using sperner lemma.We stress that the Gale-Nikaido-Debreu lemma can be established as a corollary of the Kakutani theorem. This is demonstrated by modifying Uzawa's (1962) argument[4] for continuous mapping. Some authors have tried to use the Sperner lemma to prove the Kakutani Theorem. Sondjaja (2008)[5] uses the Sperner lemma to produce a proof, but he must also employ the approximation lemma of von Neumann (1937)[6]. The proof becomes somewhat more difficult as a result. Tanaka (2012)[7] generalizes Sperner's original lemma by proving the so-called hyperplane labeling lemma. He then proves the Kakutani Theorem by combining this conclusion with the approximate minimax theorem. Since our demonstration merely makes use of the fundamental ideas of combinatorial topology, it appears to be clearer. We go over some fundamental ideas in Section 2, including the Sperner lemma, simplicial subdivision, and the idea of a subsimplex. In Section 3, we show the proof of Kakutani fixed point theorem and the Gale-Nikaido-Debreu lemma using the Sperner lemma. Lastly, the paper is concluded in Section 4.

2 Preliminaries

This part provides the background information and fundamental terms needed for our work. Prior to stating the Sperner lemma, we provide some definitions related to the further work. After that we have demonstrated the proof of fixed point theorem using well known sperners lemma .we have proved Gale-

Nikaido-Debreu lemma using Sperners lemma.Moreover it is also shown that Gale-Nikaido-Debreu lemma can be established as a corollary of the Kakutani theorem.

2.1 On Sperner lemma

Let R^n be the Euclidean space. Let the n unit vectors of R^n be represented by $e^1 = (1,0,0,\dots,0), e^2 = (0,1,0,\dots,0),\dots, and\ e^n = (0,0,\dots,0,1)$. The convex hull of $\{e^1, e^2,\dots,e^n\}$ is the unit-simplex Δ of R^n. The convex hull of $\{x^1, x^2,\dots,x^n\}$ where $x^i \in \Delta$ for any $i = 1,2,\dots,n$, is a simplex of Δ,represented by $[[x^1, x^2,\dots,x^n]]$ and the vectors $(x^1 - x^2, x^1 - x^3, \dots, x^1 - x^n)$ are linearly independent. Given a simplex $[[x^1, x^2,\dots,x^n]]$, a face of this simplex is the convex hull $[[x^{i_1}, x^{i_2}, \dots, x^{i_m}]]$ with $m < n$, and $\{i_1, i_2,\dots,i_m\} \subset \{1,2,\dots,n\}$.

The concepts of labeling and simplicial subdivision (also known as triangulation) are now defined (Border (1985)[8] and Su (1999)[9] for a general treatment) before stating the Sperner lemma.

Definition 1. S is a simplicial subdivision (triangulation) of Δ if it is a finite collection of simplices and their faces $\Delta_i, i = 1,2,\dots,q$ such that

- $\Delta = U_{i=1}^{q} \Delta_i$
- $r_i(\Delta_i) \cap r_i(\Delta_j) = \emptyset, \forall\ i \neq j.$

Remember if $\Delta_i = [x^{i_1}, x^{i_2}, \dots, x^{i_m}]$, then we have

$ri(\Delta_i) \equiv \{x | x = \sum_{k=1}^{m} \gamma_x x^k\ (i); \sum_k \gamma_k = 1; and\ \forall k{:}\, \gamma(k) > 0\}$.

Simply, we can say that a simplicial subdivision divides an n-dimensional simplex into smaller simplices so that any two simplices share a full face of a given dimension or are disjoint.

Definition 2 . Consider S a simplicial subdivision of Δ. Let V denote the set of vertices of all the subsimplices of Δ. Now, a labeling R is a function from V into $\{1,2,\dots,n\}$. A labeling R satisfies the Sperner condition if:

$$x \in ri[\![e^{i_1}, e^{i_2}, \dots, e^{i_m}]\!] \Rightarrow R(x) \in \{i_1, i_2, \dots i_m\}$$

Particularly, $R(e^i) = i, \forall i$.

We must keep in mind that all of the vertices of the simplex must have distinct labels according to the Sperner condition. Furthermore, any vertex on the edge connecting the original simplex's vertices has a label that corresponds to another label for these vertices. Keeping these in mind, we can now state the Sperner lemma.

Lemma 1. (Sperner) Let a simplicial subdivision of Δ be $S = \{\Delta_1, \Delta_2, \dots, \Delta_q\}$. Let R be a labeling that satisfies the Sperner condition. Then there exists a completely labeled subsimplex $\Delta_i \in S$, i.e. $\Delta_i = [[x^1(i), x^2(i), \dots, x^n(i)]]$ with $R(x^l(i)) = l, \forall\ l = 1,2,\dots,n$.

According to the Sperner condition, the presence of a completely labeled subsimplex for each simplicially subdivided simplex is guaranteed by the Sperner lemma. Numerous textbooks and studies, including Sperner (1928)[10], Berge (1959)[11], Scarf and Hansen (1973)[12], and Le Van (1982)[13], provide proofs of this lemma. For a set of lower dimensional problems, the original proof specifically employs an inductive argument based on a comprehensive list of all completely labeled simplices. Constructive arguments have been used in proofs since Cohen (1967)[14] and Kuhn (1968)[15] (for an example of the constructive proof, Scarf (1982)[16].

2.2 On correspondences

Suppose $X \subset R^l, Z \subset R^m$. A mapping T from X into Z is a mapping from X into the set of subsets of Z. The graph of T is the set graph $T = \{(x, z) \in X \times Z : z \in T(x)\}$.

A mapping $T: X \to Z$ is closed if its graph is closed.

Definition 3. A mapping $T : X \to Z$ is lower semicontinuous at point x if for any $z \in T(x)$ and for any sequence $\{x^n\} \subset X$ converging to x, there exists a subsequence $\{z^{n_k}\}$ with $z^{n_k} \in T(x^{n_k})$, $\forall k$, such that $\{z^{n_k}\}$ converges to z when k converges to $+\infty$. T is lower semicontinuous on X if it is lower semicontinuous everywhere on X.

Definition 4. A mapping $T : X \to Z$ is upper semicontinuous at point x if (i) $T(x)$ is compact, non-empty, and (ii) for any sequence $\{x_n\}$ converging to x, for any sequence $\{z_n\}$ with $z_n \in T(x_n)$, ∀n, there exists a subsequence $\{z_{n_k}\}$ which converges to $z \in T(x)$.

A mapping is continuous if it is both lower semicontinuous and upper semi- continuous. Note that if X is compact then T is upper semicontinuous if and only if T is closed. It is also clear that if T is upper semicontinuous and $K \subset X$ is compact, then $T(K)$ is compact. Recall that if T is single valued, the notions of continuity, upper semicontinuity, and the lower semicontinuity turn out to be equivalent.

3 Main results

3.1 Proof of fixed point theorem using Sperner lemma:

The Brouwer fixed point theorem is regarded as one of most fundamental findings. The Brouwer theorem is extended for the case of set-valued functions by the Kakutani fixed point theorem. These two theorems are widely applicable in many areas of economics and mathematics.

The Kakutani theorem is now formally stated, and it is proved using the Sperner lemma.

Theorem 1. (Kakutani) Let ψ be an upper semi continuous mapping, with non empty convex compact values from a non-empty convex, compact set $V \subset R^N$ into itself. Then there exists a fixed point x, i.e. $x \in \psi(x)$.

Proof. Here, we provide a proof for the case where the set V is the unit-simplex Δ of R^N, since any convex compact set in R^N is homeomorphic to a simplex.

Let $\epsilon > 0$ be given. Given that Δ is compact, a finite covering of Δ has a finite family of open balls $\left(\tilde{B}(x^i(\epsilon), \epsilon)\right)_{i=1,2,\ldots,I(\epsilon)}$. Take a partition of unity that is subordinate to the family $\left(\tilde{B}(x^i(\epsilon), \epsilon)\right)_{i=1,2,\ldots,I(\epsilon)}$, i.e. a family of continuous non-negative real functions $(\alpha_i)_{i=1,2,\ldots,I(\epsilon)}$ from Δ in R_+ such that

$$Supp(\alpha_i) \subset B\left(x^i(\epsilon), \epsilon\right), \forall i \text{ and } \sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1, \forall x \epsilon \Delta.$$

Let us Take $y^i(\epsilon) \in \psi\left(x^i(\epsilon)\right), \forall i$ and define a continuous function $g^\epsilon: \Delta \to \Delta$ by

$$g^{\epsilon}(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon).$$

Consider a simplicial subdivision T^K where $k > 0$ be an integer such that $Mesh(T^K) < \frac{1}{K}$. Now, define a labeling R as follows:

for $x \in \Delta, R(x) = l, if\ x_l \geq g_l^{\epsilon}(x).$ (1)

Such a labeling is well defined because $\sum_l x_l = \sum_l g_l^{\epsilon}(x) = 1$. Additionally Sperner condition is followed by this labelling. Now take $x \in ri[\![e^{i_1}, e^{i_2}, \ldots, e^{i_r}]\!]$,where $(e^i)_i$ are the unit vectors of R^N. We claim that $R(x) \in \{i_1, i_2, \ldots, i_r\}$. If not ,$x_l < g_l^{\epsilon}(x), \forall i \in \{i_1, i_2, \ldots, i_r\}$,we get a contradiction:

$1 = \sum_{l \in \{i_1, i_2, \ldots, i_r\}} x_l < \sum_{l \in \{i_1, i_2, \ldots, i_r\}} g_l^{\epsilon}(x) \leq 1.$

As per sperner lemma, ∃ a completely labelled subsimplex $S^K = [\![x^{K,1}, x^{K,2}, \ldots, x^{K,N}]\!]$ with $x_l^{K,l} \geq g_l^{\epsilon}(x^{k,l}) \forall l = 1,2, \ldots, N$.

Let $K \to +\infty$, there exixts a subsequence $(K_t)_{t \geq 1}$such that $x^{K_t,l}$ converges to x^l for any $l = 1,2, \ldots, N$. Since $Mesh(T^K) \to 0$,we must have $x^1 = x^2 = \cdots = x^N$.

Let $x^*(\epsilon)$ be this point. From continuity, we have $g^{\epsilon}(x^{K_t,l}) \to g^{\epsilon}(x^*(\epsilon)) \forall l$. Since

$x_l^*(\epsilon) \geq g_l^{\epsilon}(x^*(\epsilon)) \forall l$, we have $x^*(\epsilon) = g^{\epsilon}(x^*(\epsilon))$.

Since $(\tilde{B}(x^i(\epsilon), \epsilon))_{i=1,2,\ldots,I(\epsilon)}$ is a covering of Δ, we have $x^*(\epsilon) \in \cap_{i \in J(\epsilon)} \tilde{B}(x^i(\epsilon), \epsilon)$,

Where $J(\epsilon) \subset \{1,2, \ldots, I(\epsilon).\}$.

Hence,

$x^*(\epsilon) = g^{\epsilon}(x^*(\epsilon)) = \sum_{i \in J(\epsilon)} \alpha^i(x^*(\epsilon)) y^i(\epsilon)$ (2.1)

with $\sum_{i \in J(\epsilon)} \alpha^i(x^*(\epsilon)) = 1, y^i(\epsilon) \in \psi\left(x^i(\epsilon)\right), \forall i \in J(\epsilon)$. (2.2)

Observe that $\forall i \in J(\epsilon), x^i(\epsilon) \in B(x^*(\epsilon), \epsilon) \subset R^N$. Therefore, $y^i(\epsilon) \in \psi\,(B(x^*(\epsilon), \epsilon))$

And $g^{\epsilon}(x^*(\epsilon)) \in co(\psi\,(B(x^*(\epsilon), \epsilon)))$.

According to Caratheodory convexity theorem, we have a decomposition

$g^{\epsilon}(x^*(\epsilon)) = \sum_{i=1}^{N+1} B_i(x^*(\epsilon)\, \tilde{y}^i(x^*(\epsilon))$ (3)

With $\tilde{y}^i(x^*(\epsilon)) \in \psi\,(B(x^*(\epsilon), \epsilon)), \beta_i(x^*(\epsilon), \geq 0, \sum_{i=1}^{N+1} \beta_i(x^*(\epsilon)) = 1$.

Let $\epsilon \to 0$, Suppose (without loss of generality) $x^*(\epsilon) \to \bar{x} \in \Delta, \beta_i(x^*(\epsilon)) \to \bar{\beta}_i \geq 0, \sum_{i=1}^{N+1} \bar{\beta}_i = 1$, and $\tilde{y}^i(x^*(\epsilon)) \to \bar{y}^i \in \psi\,(\bar{x}), \forall i = 1,2, \ldots, N+1$.

This implies $\bar{x} = \sum_{i=1}^{N+1} \bar{\beta}_i \bar{y}^i$. Since $\psi\,(\bar{x})$ is convex,we get $\bar{x} \in \psi\,(\bar{x})$.This concludes the proof.

Corollary of kakutani fixed point theorem stated below is Brouwer fixed point theorem when ψ is a single valued mapping.

Corollary 1: (Brouwer) Let σ represent a continuous mapping into itself from a non-empty convex compact set. Then $x = \sigma(x)$, which is a fixed point, exists.

3.2 Using Sperner lemma to prove Gale-Nikaido-Debreu lemma

we show that GND lemma can be proved by using only Sperner lemma.

Lemma 2. (Gale-Nikaido-Debreu lemma) Let Δ be the unit-simplex of R^N. Let ψ be a continuous mapping from Δ *into* R^N. Suppose ψ satisfies the condition

$$\forall q \in \Delta, q \cdot \psi(q) \leq 0.$$

Then there exists $\bar{q} \in \Delta$ such that $\psi(\bar{q}) \leq 0$.

Proof. Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of R^N such that $Mesh(T^K) < 1/K$. With any vertex q^i *of* T^K, we associate $\psi(q^i)$. We have $q^i \cdot \psi(q^i) \leq 0$. We consider the following labelling :

$$\text{For } q \in \Delta, R(q) = i \text{ if } \psi_i(q) \leq 0.$$

This labeling is well defined. Indeed, if not, $\psi_i(q) > 0, \ \forall i$ *and* $0 \geq \sum_i q_i \psi_i(q) > 0$ leads to a contradiction. Note that the Sperner condition is satisfied by this labeling.

Now, take $q \in ri[[e^{i_1}, e^{i_2}, \ldots, e^{i_m}]], m < N$. Then $R(q) \in \{i_1, i_2, \ldots, i_m\}$. If not, $\psi_i(q) > 0, \forall i \in \{i_1, i_2, \ldots, i_m\}$ and $0 \geq q.\psi(q) = \sum_{i \in \{i_1, i_2, \ldots, i_m\}} q_i \psi_i(q) > 0$. That is a contradiction.

Now, for any K ,there exists a completely labelled subsimplex $[[q^{i_1}, q^{i_2}, \ldots, q^{i_N}]]$ as per sperner lemma which satisfies for any $l = 1,2, \ldots, N, \psi_l(q^{i_l}) \leq 0$. Let $K \to +\infty$. Then, $\forall l, q^{i_l} \to \bar{q}$ and $\psi(q^{i_l}) \to \psi(\bar{q})$. Obviously, $\psi_l(\bar{q}) \leq 0, \forall l = 1,2, \ldots, N$.

In other words, $\psi(\bar{q}) \leq 0$. Additionally, we may examine the Gale-Nikaido-Debreu lemma in both its strong and alternate forms, and crucially, we demonstrate that either version can be effectively proved using Sperner lemma.

Lemma 3. (Gale-Nikaido-Debreu lemma: strong version) Let Δ be the unit-simplex of R^N. Let ψ be an upper semi-continuous correspondence with non-empty, compact, convex values from Δ *into* R^N. Suppose ψ satisfies the following condition:

$$\forall q \in \Delta, \forall z \in \psi(q), q \cdot z \leq 0. \quad (4)$$

Then there exists $\bar{q} \in \Delta$ such that $\psi(\bar{q}) \cap R^N \neq \emptyset$.

Proof. Let $P = max\{\|z\|_1 : z \in \psi(\Delta)\}$. Let $\epsilon \in (0,1)$. Since Δ is compact, there exists a finite covering of Δ with a finite family of open balls $\left(\tilde{Q}(x^i(\epsilon), \epsilon)\right)_{i=1,2,\ldots,I(\epsilon)}$.

Take a partition of unity subordinate to the family $\left(\tilde{Q}(x^i(\epsilon), \epsilon)\right)_{i=1,2,\ldots,I(\epsilon)}$, i.e. a family of continuous non negative real functions $(\alpha_i)_{i=1,2,\ldots,I(\epsilon)}$ from Δ in R_+ such that Supp $\alpha_i \subset Q\left(x^i(\epsilon), \epsilon\right), \forall i$ and $\sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1 \ \forall x \in \Delta$. Take $y^i(\epsilon) \in \psi(x^i(\epsilon)), \forall i$ and define the function $g^\epsilon(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon) \in \Delta$.

This function is continuous.

Given $x \in \Delta$, there exists a set $J(x) \subset \{1,2, \ldots, I(\epsilon)\}$ such that $x \in \cap_{i \in J(x)} \tilde{Q}\left(x^i(\epsilon), \epsilon\right)$.

We have $g^\epsilon(x) = \sum_{i \in J(x)} \alpha_i(x) y^i(\epsilon)$ with $\sum_{i \in J(x)} \alpha_i(x) = 1$.We have

$\forall i \in J(x), x^i(\epsilon) = x + \epsilon u^i(x)$, with some $u^i(x) \in Q(0,1)$ which implies that:

$\forall i \in J(x), y^i(\epsilon) \in \psi\left(x^i(\epsilon)\right) = \psi(x + \epsilon u^i(x)) \subset \psi(Q(x, \epsilon))$.

By consequence, $g^\epsilon(x) \in co\left(\psi\left(Q(x, \epsilon)\right)\right)$.

According to Caratheodory's convexity theorem, we have a decomposition $g^{\epsilon}(x) = \sum_{i=1}^{N+1} \beta_i (x,\epsilon)\tilde{y}^i(x,\epsilon)$

With $\tilde{y}^i(x,\epsilon) \in \psi(x + \epsilon u^i)$ where $u^i(x) \in Q(0,1), \beta_i(x,\epsilon) \geq 0, \sum_{i=1}^{N+1} \beta_i (x,\epsilon) = 1$. From this, we have $x.g^{\epsilon}(x) = \sum_{i=1}^{N+1} \beta_i (x,\epsilon)\left(x + \epsilon u^i\right).\tilde{y}^i(x,\epsilon) - \epsilon \sum_{i=1}^{N+1} \beta_i (x,\epsilon)u^i.\tilde{y}^i$

$$\leq \epsilon \sum_{i=1}^{N+1} \beta_i (x,\epsilon)\|u^i\|.\|\tilde{y}^i\| \leq \epsilon P \sum_{i=1}^{N+1} \beta_i (x,\epsilon) = \epsilon P$$

Since $\left(x + \epsilon u^i\right).\tilde{y}^i(x,\epsilon) \leq 0$ (see equation 4), $\|u^i\| \leq 1$ *and* $\|\tilde{y}^i\| \leq P$. Therefore, we get that

$$\forall x \in \Delta, \exists i, g_i^{\epsilon}(x) \leq \epsilon P. \qquad (5)$$

Indeed, if $\forall i, g_i^{\epsilon}(x) > \epsilon P$, then $\epsilon P < \sum_i x_i\, g_i^{\epsilon}(x)) \leq \epsilon P$.

Which is a contradiction.

Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of R^N such that $Mesh(T^K) < 1/K$ and define the labelling R as follows:

$$\forall x \in \Delta, R(x) = i, if\, g_i^{\epsilon}(x) \in (x) \leq \epsilon\, P.$$

This labeling is well-defined according to (5) and also satisfies the Sperner condition

$$x \in [[e^{i_1}, e^{i_2}, \dots, e^{i_m}]] \Rightarrow R(x) = i \in \{i_1, i_2, \dots, i_m\}$$

Indeed,if $g_i^{\epsilon}(x) > \epsilon P, \forall i \in \{i_1, i_2, \dots, i_m\}$, then $\epsilon P \geq x.g^{\epsilon}(x) = \sum_{i \in \{i_1,i_2,\dots,i_m\}} x_i g_i^{\epsilon}(x) > \epsilon \sum_{i \in \{i_1,i_2,\dots,i_m\}} x_i = P\epsilon,$

which is a contradiction.

Sperner lemma implies that there exists a completely labelled sub-simplex $[[x^{K,1}, x^{K,2}, \dots, x^{K,N}]]$ with $R(x^{K,l}) = l, \forall l = 1,2, \dots, N, i.e\; g_l^{\epsilon}(x^{K,l}) \leq \epsilon P, \forall l = 1,2, \dots, N.$

Let $K \to +\infty$, there is a subsequence (K_t) such that

$$\forall l, x^{K_t,l} \to x^{\epsilon} \epsilon \Delta, g^{\epsilon}(x^{K_t,l}) \to g^{\epsilon}(x^{\epsilon})$$

and, therefore, $g_l^{\epsilon}(x^{\epsilon}) \leq \epsilon P, \forall l = 1,2, \dots, N.$

Since $\left(\tilde{Q}(x^i(\epsilon),\epsilon)\right)_{i=1,2,\dots,I(\in)}$ is a covering of Δ, there exists a set $J(x^{\epsilon}) \subset \{1,2, \dots, I(\in)\}$ such that $x \in \cap_{i \in J(x^{\epsilon})} \tilde{Q}(x^i(\epsilon),\epsilon)$.

We have $g^{\epsilon}(x^{\epsilon}) = \sum_{i \in J(x^{\epsilon})} \alpha_i (x^{\epsilon}) y^i(x^{\epsilon})$ with $\sum_{i \in J(x^{\epsilon})} \alpha_i (x^{\epsilon}) = 1$.

Use the same argument as in our proof of Kakutani theorem, we get

$$g^{\epsilon}(x^{\epsilon}) = \sum_{i=1}^{N+1} \beta_i(x^{\epsilon})\tilde{y}^i(x^{\epsilon})$$

with $\tilde{y}^i(x^{\epsilon}) \in \psi\left(Q(x^{\epsilon},\epsilon)\right)$.

Let $\epsilon \to 0$, we get that

$x^{\epsilon} \to \bar{x} \in \Delta,\; \beta_i(x^{\epsilon}) \to \bar{\beta}_i \geq 0, \sum_{i=1}^{N+1} \bar{\beta}_i = 1,$

$$\tilde{y}^i(x^{\epsilon}) \to \bar{y}^i \in \psi(\bar{x}), \forall i = 1,2, \dots, N+1$$

$g^{\epsilon}(x^{\epsilon}) \to \bar{z} = \sum_{i=1}^{N+1} \bar{\beta}_i \bar{y}^i \in \psi(\bar{x})$, since $\psi(\bar{x})$ is convex

$g_l^{\epsilon}(x^{\epsilon}) \leq \epsilon A, \forall l = 1,2, \dots, N$ which implies $\bar{z}_l \leq 0, \forall l = 1,2, \dots, N.$

This implies $\bar{z} \in \psi(\bar{x}) \cap R_{-}^{N}$. This concludes the proof.

We may also obtain two more robust versions of the GND lemma from Lemma 3. Below are statements and proofs for each.

Lemma 4. Let Δ be the unit-simplex of R^N. Let ψ be an upper semicontinuous correspondence with non-empty, compact, convex values from Δ into R^N. Suppose ψ satisfies the condition

$$\forall p \in \Delta, \forall z \in \psi(p), p \cdot z = 0.$$

Then there exists $\bar{p}, \bar{z} \in \psi(\bar{p})$ such that (1) $\bar{z} \leq 0$, and (2) $\forall i = 1, \dots, N, \bar{p}_i \neq 0 \Rightarrow \bar{z}_i = 0$.

Proof. Since "$\forall p \in \Delta, \forall z \in \psi(p), p \cdot z = 0$" $\Rightarrow$ "$\forall p \in \Delta, \forall z \in \psi(p), p \cdot z \leq 0$", so ,there exists $\bar{p}$ *and* $\bar{z} \in \psi(\bar{p})$ (from Lemma 3) such that $\bar{z} \leq 0$. Since $\bar{p}.\bar{z} = 0$, The outcome is instantaneous..

Lemma 5. Let Δ be the unit-simplex of R^N. Let ψ be an upper semicontinuous correspondence with non-empty, compact, convex values from Δ into R^N. Suppose ψ satisfies the condition $\forall p \in \Delta, \exists z \in \psi(p), p \cdot z \leq 0$.

Then there exists $\bar{p} \in \Delta$ such that $(\bar{p}) \cap R_-^N \neq \varphi$.

Proof. For $p \in \Delta$, let $\tilde{\psi}(p) = \{z \in \psi(p): z \cdot p \leq 0\}$. The correspondence $\tilde{\psi}$ is upper semicontinuous, compact valued and convex from Δ into R^N. It satisfies the assumptions of Lemma 3. Hence there exist $\bar{p}$ *and* $\bar{z} \in \tilde{\psi}(\bar{p}) \subset \psi(\bar{p})$, such that $\bar{z} \leq 0$.

Lemma 2 provides a direct demonstration for the alternate statement of the GND lemma that we now examine.

Lemma 6. (Gale-Nikaido-Debreu) Let S denote the unit-sphere, for the norm $\|.\|_2$ of R^N.

Let ψ be an upper semicontinuous mapping from $S \cap R_+^N$ in R^N that satisfies $\forall q \in S \cap R_+^N, \forall z \in \psi(q), q.z \leq 0$.

Then there exist $\bar{q} \in S \cap R_+^N$, such that $\psi(\bar{q}) \cap R_-^N \neq \varphi$.

Proof. For $p \in \Delta$ define $\chi(p) = \frac{1}{\sqrt{p_1^2+p_2^2+\cdots+p_N^2}}$, and for $q \in S \cap R_+^N$, define $\rho(q) = \sum_{i=1}^N q_i$.

We have,if $q \in S \cap R_+^N$ then $p = \frac{q}{\rho(q)} \in \Delta$ and if $p \in \Delta$ then

$q = \chi(p)p \in S \cap R_+^N$. Define also for $p \in \Delta$, $\eta(p) = \psi(\chi(p)p)$. Obviously, η is upper semicontinuous with convex and compact values. We have

$$\forall p \in \Delta, \forall z \in \eta(p) = \psi(\chi(p)p), \mu(p)p \cdot z \leq 0 \Leftrightarrow p \cdot z \leq 0.$$

From Lemma 2, There Exist $\bar{p} \in \Delta$ and $\bar{z} \in \eta(\bar{p})$ Such that $\bar{z} \leq 0$ or, equivalently, there exist $\bar{q} = \chi(\bar{p})\bar{p}, \bar{z} \in \eta(\bar{p}) = \psi(\mu(\bar{p})\bar{p}) = \psi(\bar{q})$ such that $\bar{z} \leq 0$.

Theorem 3: Kakutani theorem is a corollary of Gale-Nikaido-Debreu lemma.

Proof: let ψ be an upper semicontinuous mapping with non-empty convex compact values from Δ into itself. Define for $q \in \Delta$,

$$\mu(q) = \{y: y = z - \frac{q.z}{\sum_{i=1}^N \bar{q}_i^2} q, with\ z \in \psi(q)\}$$

Clearly, μ is upper semicontinuous and convex valued. Also for any $q \in \Delta$, any $y \in \mu(q)$,

We have $q.y = 0$.Hence from lemma 4,there exist $\bar{q} \in \Delta$ and $\bar{y} \in \mu(\bar{q})$ that satisfy $\bar{y} \leq 0$, and $\forall i = 1,2, \dots, N$, $\bar{q}_i \neq 0 \Rightarrow \bar{y}_i = 0$. In other words,there exist $\bar{q} \in \Delta$ and $\bar{z} \in \psi(\bar{q})$ satisfying following two conditions:

$$\forall i = 1,2, \dots, N, \bar{z}_i \leq \frac{\bar{q}.\bar{z}}{\sum_{i=1}^N \bar{q}_i^2} \bar{q}_i$$

$\forall i = 1,2, \dots, N, \bar{q}_i \neq 0 \Rightarrow \ \bar{z}_i \leq \frac{\bar{q}.\bar{z}}{\sum_{i=1}^{N} \bar{q}_i^2} \bar{q}_i.$

Hence, if $\bar{q}_i = 0$, we have $0 \leq \bar{z}_i \leq 0 \ \Rightarrow \bar{z}_i = 0$.

Let $$\varphi = \frac{\bar{q}.\bar{z}}{\sum_{i=1}^{N} \bar{q}_i^2}.$$

We obtain that $\bar{z}_i = \varphi \bar{q}_i$ for any $i = 1,2, \dots, N$. Since $\bar{z} \in \Delta, \bar{q} \in \Delta$,we have $\varphi = 1$. Hence ,$\bar{q} = \bar{z} \in \psi(\bar{q})$.

4 Conclusion

We have demonstrated in this research that the Sperner lemma is an effective tool for learning a few ideas.Using the Sperner lemma, we have demonstrated both the Gale-Nikaido-Debreu lemma and the Kakutani theorem.We believe that our research offers a significant first step in bolstering the novel application of the Sperner lemma.

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