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(S, d) Magic Labeling of Trees of Diameter Less Than or Equal to Four

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Abstract: Let G (p, q) be a connected, undirected, simple and non-trivial graph with p vertices and q edges. Let f be an injective function f: V(G) → $\{s, s+d, s+2d, \ldots\ldots s+(q+1)d\}$ and g be an injective function g: E(G) → $\{d, 2d, 3d \ldots\ldots 2(q-1)d\}$.Then the graph G is said to be (s, d) magic labeling if $f(u) + g(uv) + f(v)$ is a constant, for all $u, v \in V(G)$. A graph G is called (s, d) magic graph if it admits (s, d) magic labeling

Index Terms - Tree, Diameter, Graph labeling, (S, d) Magic Labeling.

I. Introduction

In this paper, we focus exclusively on simple, connected, and non-trivial graphs $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The number of vertices and edges in G are denoted by $|V(G)| = p$ and $|E(G)| = q$, respectively.

The concept of graph labeling has gained significant recognition within graph theory due to its broad applications in various fields. Graph labeling techniques play a crucial role in developing mathematical models for diverse purposes, including X-ray crystallography, coding theory, radar systems, cryptography, communication network design, astronomy, circuit layout optimization, and database management.

A tree is defined as a connected graph that contains no cycles. For a connected graph $G(V, E)$, the distance $d(u, v)$ between two vertices u and v represents the minimum length of a path connecting them. The eccentricity $e(v)$ of a vertex v is the maximum distance from v to any other vertex in the graph. The diameter of a graph G is the largest eccentricity among all vertices.

In this paper the existence of (s, d) magic labeling of trees T_1^2 of diameter two, trees of diameter three denoted by T_s^3 for $1 \leq s \leq 4$ and trees of diameter four denoted by T_s^4 for $1 \leq s \leq 4$ are analysed. Further, it has been proved that these trees admit (S, d) magic labeling.

II.NOTATION

Notation 2.1: T_1^2 – the tree of diameter two obtained by attaching n pendant edges to the internal vertex of the path P_3.

Notation 2.2: The trees of diameter 3 denoted by T_s^3, $1 \le s \le 4$ are defined as follows.

1. T_1^3 – the tree of diameter three obtained by attaching n pendant edges to the first internal vertex of the path P_4.
2. T_2^3 – the tree of diameter three obtained by attaching n pendant edges to the second internal vertex of the path P_4.
3. T_3^3 – the tree of diameter three obtained by attaching m, n pendant edges to the first and second internal vertex of the path P_4.
4. T_4^3 – the tree of diameter three obtained by attaching n pendant edges through an edge to the internal vertex of the path P_3.

Notation 2.3: The trees of diameter 4 denoted by T_s^4, $1 \le s \le 4$ are defined as follows.

1. T_1^4 – the tree of diameter four obtained by attaching n pendant edges to the first internal vertex of the path P_5.
2. T_2^4 – the tree of diameter four obtained by attaching n pendant edges to the second internal vertex of the path P_5.
3. T_3^4 – the tree of diameter four obtained by attaching n pendant edges to the third internal vertex of the path P_5.
4. T_4^4 – the tree of diameter four obtained by attaching m, n pendant edges to the first and second internal vertex of the path P_5.

III.MAIN RESULTS

3.1 Trees of Diameter Two

Theorem 3.1.1 The tree T_1^2 obtained by attaching n pendant edges to the internal vertex of the path P_3 is a (S, d) magic graph

Proof: Let $V(T_1^2) = \{t_i : 1 \le i \le 3\} \cup \{x_j : 1 \le j \le n\}$ and $E(T_1^2) = \{t_i t_{i+1} : 1 \le i \le 2\} \cup \{t_2 x_j ; 1 \le j \le n\}$

$p = n + 3$ and $q = n + 2$

Define $f : V(G) \to \{s, s + d, s + 2d, \dots . s + (q + 1)d\}$ to label the vertices as follows

$f(t_1) = s$

$f(t_2) = s + d$

$f(t_3) = s + 2d$

$f(x_j) = s + (j + 2)d; 1 \le j \le n$

Define $g : E(G) \to \{d, 2d, 3d, \dots .2(q - 1)d\}$ to label the edges as follows

$g(t_i t_{i+1}) = 2s + 2(q - 1)d - (f(t_i) + f(t_{i+1}) : 1 \le i \le 2$

$g(t_2 x_j) = 2s + 2(q - 1)d - (f(t_2) + f(x_j)) : 1 \le j \le n$

Table 3.1 Labeling of Vertices for T_1^2

Value of i and j	$f(x_j)$	$f(t_{i+1})$
$0 \leq i \leq 2$	$-$	$s + id$
$1 \leq j \leq n$	$s + (j + 2)d$	$-$

Table 3.2 Labeling of Edges for T_1^2

Value of i and j	$g(t_2x_j)$	$g(t_it_{i+1})$
$1 \leq i \leq 2$	$-$	$2s + 2(q-1)d - (f(t_i) + f(t_{i+1}))$
$1 \leq j \leq n$	$2s + 2(q-1)d - (f(t_2) + f(x_j))$	$-$

From the above table we find that f *and* g are injective and $f(t_i) + f(t_{i+1}) + g(t_it_{i+1})$ and $f(t_2) + f(x_j) + g(t_2x_j)$ are constant equal to 2(s+(q−1)d) .Hence we concluded that the T_1^2 admits (S,d) magic labeling.

Example 3.1.2 The (S ,d) magic labeling of T_1^2 is shown below

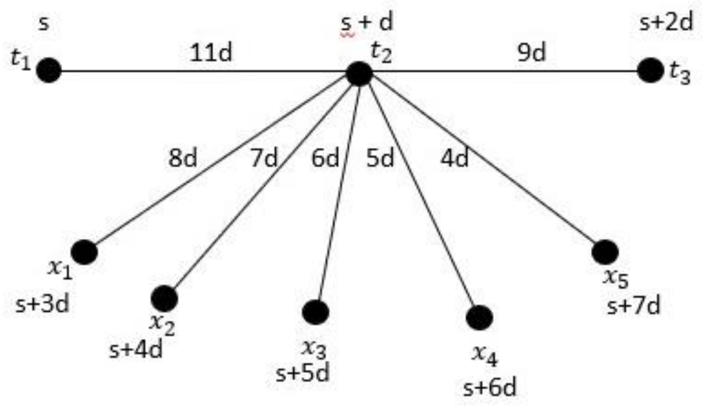

Figure 3.1 Tree of diameter 2, T_1^2

3.2 Trees of Diameter Three

Theorem 3.2.1 The tree T_1^3 obtained by attaching n pendant edges to the first internal vertex of the path P_4 is a (S, d) magic graph

Proof: Let $V(T_1^3) = \{t_i : 1 \leq i \leq 4\} \cup \{x_j : 1 \leq j \leq n\}$ and $E(T_1^3) = \{t_it_{i+1} : 1 \leq i \leq 3\} \cup \{t_2x_j ; 1 \leq j \leq n\}$

$p = n + 4$ and $q = n + 3$

Define $f: V(G) \rightarrow \{s, s + d, s + 2d, \ldots . s + (q + 1)d\}$ to label the vertices as follows

$f(t_1) = s$

$f(t_2) = s + d$

$f(t_3) = s + 2d$

$f(t_4) = x_n + d$

$f(x_j) = s + (j+2)d; 1 \le j \le n$

Define $g: E(G) \to \{d, 2d, 3d, \ldots .2(q-1)d\}$ to label the edges as follows

$g(t_i t_{i+1}) = 2s + 2(q-1)d - (f(t_i) + f(t_{i+1}) : 1 \le i \le 2$

$g(t_2 x_j) = 2s + 2(q-1)d - (f(t_2) + f(x_j) : 1 \le j \le n$

Table 3.3 Labeling of Vertices for T_1^3

$f(t_4) = x_n + d$		
Value of i and j	$f(x_j)$	$f(t_{i+1})$
$0 \le i \le 2$	$-$	$s + id$
$1 \le j \le n$	$s + (j+2)d$	$-$

Table 3.4 Labeling of Edges for T_1^3

Value of i and j	$g(t_2 x_j)$	$g(t_i t_{i+1})$
$1 \le i \le 3$	$-$	$2s + 2(q-1)d - (f(t_i) + f(t_{i+1}))$
$1 \le j \le n$	$2s + 2(q-1)d - (f(t_2) + f(x_j))$	$-$

From the above table we find that f and g are injective and $f(t_i) + f(t_{i+1}) + g(t_i t_{i+1})$ and $f(t_2) + f(x_j) + g(t_2 x_j)$ are constant equal to 2(s+(q−1)d) .Hence we concluded that the T_1^3 admits (S,d) magic labeling.

Example 3.2.2 The (S,d) magic labeling of T_1^3 is shown below

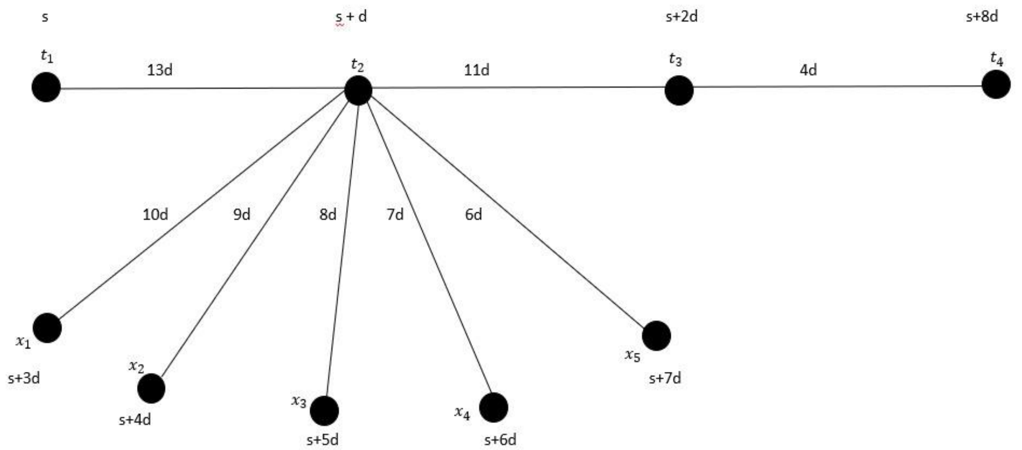

Figure 3.2 Tree of diameter 3, T_1^3

Theorem 3.2.3 The tree T_3^3 obtained by attaching m, n pendant edges to the first and second internal vertex of the path P_4 is a (S,d) magic graph

Proof: Let $V(T_3^3) = \{t_i : 1 \le i \le 4\} \cup \{x_j : 1 \le j \le n\} \cup \{y_1 : 1 \le k \le m$ and

$E(T_3^3) = \{t_i t_{i+1} : 1 \le i \le 3\} \cup \{t_2 x_j ; 1 \le j \le n\} \cup \{t_3 y_k : 1 \le k \le m\}$

$p = n + m + 4$ and $q = n + m + 3$

Define $f: V(G) \rightarrow \{s, s+d, s+2d, \dots. s+(q+1)d\}$ to label the vertices as follows

$f(t_1) = s$

$f(t_2) = s + d$

$f(t_3) = s + 2d$

$f(t_4) = y_m + d$

$f(x_j) = s + (j+2)d; 1 \leq j \leq n$

$f(y_k) = x_n + kd;\ 1 \leq k \leq m$

Define $g: E(G) \rightarrow \{d, 2d, 3d, \dots .2(q-1)d\}$ to label the edges as follows

$g(t_i t_{i+1}) = 2s + 2(q-1)d - (f(t_i) + f(t_{i+1}): 1 \leq i \leq 3$

$g(t_2 x_j) = 2s + 2(q-1)d - (f(t_2) + f(x_j): 1 \leq j \leq n$

$g(t_3 y_k) = 2s + 2(q-1)d - (f(t_3) + f(y_k): 1 \leq k \leq m$

Table 3.5 Labeling of Vertices for the tree T_3^3

$f(t_4) = y_m + d$			
Value of j and k	$f(x_j)$	$f(y_k)$	$f(t_{i+1})$
$1 \leq j \leq n$	$s + (j+2)d$	$-$	$-$
$1 \leq k \leq m$	$-$	x_n+kd	$-$
$0 \leq i \leq 2$	$-$	$-$	$s + id$

Table 3.6 Labeling of Edges for the tree T_3^3

Value of i, j&k	$g(t_i t_{i+1})$	$g(t_2 x_j)$	$g(t_3 y_k)$
$1 \leq i \leq 3$	$2s + 2(q-1)d - (f(t_i) + f(t_{i+1}))$	$-$	$-$
$1 \leq j \leq n$	$-$	$2s + 2(q-1)d - (f(t_2) + f(x_j))$	$-$
$1 \leq k \leq m$	$-$	$-$	$2s + 2(q-1)d - (f(t_3) + f(y_k))$

From the above table we find that f *and* g are injective and $f(t_i) + f(t_{i+1}) + g(t_i t_{i+1})$ and $f(t_2) + f(x_j) + g(t_2 x_j)$ and $f(t_3) + f(y_k) + g(t_3 y_k)$ are constant equal to 2(s+(q−1)d) .Hence we concluded that the tree T_3^3 admits (S,d) magic labeling.

Example 3.2.4 The (S, d) magic labeling of T_3^3 is shown below

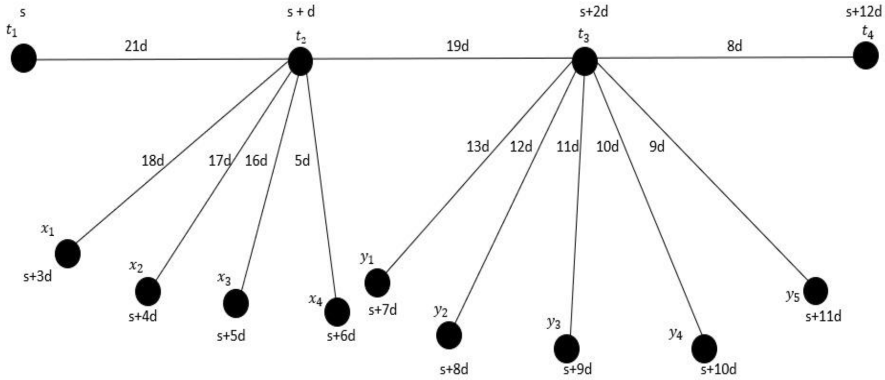

Figure 3.3 Tree of diameter 3, T_3^3

Theorem 3.2.5 The tree T_4^3 obtained by attaching n pendant edges through an edge to the internal vertex of the path P_3 is a (S, d) magic graph

Proof: Let $V(T_4^3) = \{t_i : 1 \leq i \leq 3\} \cup \{x_j : 1 \leq j \leq n\} \cup$ {x} and

$E(T_4^3) = \{t_i t_{i+1} : 1 \leq i \leq 2\} \cup \{t_2 x\} \cup \{x x_j : 1 \leq j \leq n\}$

$p = n + 4$ and $q = n + 3$

Define $f: V(G) \rightarrow \{s, s + d, s + 2d, \ldots . s + (q + 1)d\}$ to label the vertices as follows

$f(t_1) = s$

$f(t_2) = s + 2d$

$f(t_3) = x_n$+d

$f(x) = s + d$

$f(x_j) = s + (j + 2)d; 1 \leq j \leq n$

Define $g: E(G) \rightarrow \{d, 2d, 3d, \ldots .2(q - 1)d\}$ to label the edges as follows

$g(t_i t_{i+1}) = 2s + 2(q - 1)d - (f(t_i) + f(t_{i+1}) : 1 \leq i \leq 2$

$g(t_2 x) = 2s + 2(q - 1)d - (f(t_2) + f(x)$

$g(x x_j) = 2s + 2(q - 1)d - (f(x) + f(x_j) : 1 \leq j \leq n$

Table 3.7 Labeling of Vertices for the tree T_4^3

$f(x) = s + d$		
$f(t_3) = x_n + d$		
Value of i and j	$f(t_{i+1})$	$f(x_j)$
$0 \leq i \leq 2$	$s + 2id$	$-$
$1 \leq j \leq n$	$-$	$s + (j + 2)d$

Table 3.8 Labeling of Edges for the tree T_4^3

Value of i and j	$g(t_i t_{i+1})$	$g(xx_j)$
$1 \le i \le 2$	$2s + 2(q-1)d - (f(t_i) + f(t_{i+1}))$	$-$
$1 \le j \le n$	$-$	$2s + 2(q-1)d - (f(x) + f(x_j))$

From the above table we find that f *and* g are injective and $f(t_i) + f(t_{i+1}) + g(t_i t_{i+1})$ and $g(t_2 x) + f(t_2) + f(x)$ and $f(x) + f(x_j) + g(xx_j)$ are constant equal to 2(s+(q−1)d) .Hence we concluded that the tree T_4^3 admits (S,d) magic labeling.

Example 3.2.6 The (S,d) magic labeling of T_4^3 is shown below

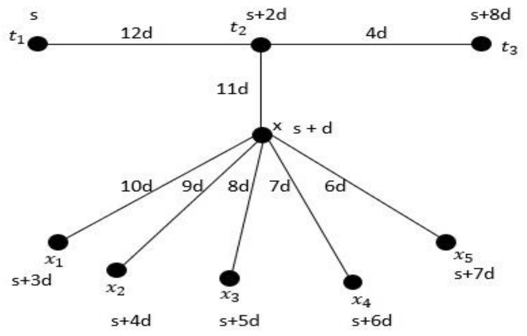

Figure 3.4 Tree of diameter 3, T_4^3

3.3 Trees of diameter four

Theorem 3.3.1 The tree T_1^4 obtained by attaching n pendant edges to the first internal vertex of the path P_5 is a (S,d) magic graph.

Proof: Let $V(T_1^4) = \{t_i : 1 \le i \le 5\} \cup \{x_j : 1 \le j \le n\}$ and

$E(T_1^4) = \{t_i t_{i+1} : 1 \le i \le 4\} \cup \{t_2 x_j ; 1 \le j \le n\}$

$p = n + 5$ and $q = n + 4$

Define $f: V(G) \to \{s, s+d, s+2d, \dots . s+(q+1)d\}$ to label the vertices as follows

$f(t_1) = t_n$+d

$f(t_2) = s$

$f(t_3) = s + d$

$f(t_4) = x_n$+d

$f(t_5) = x_n$+2d

$f(x_j) = s + (j+1)d; 1 \le j \le n$

Define $g: E(G) \to \{d, 2d, 3d, \dots .2(q-1)d\}$ to label the edges as follows

$g(t_i t_{i+1}) = 2s + 2(q-1)d - (f(t_i) + f(t_{i+1}): 1 \le i \le 4$

$g(t_2 x_j) = 2s + 2(q-1)d - (f(t_2) + f(x_j): 1 \le j \le n$

Table 3.9 Labeling of Vertices for the tree T_1^4

$f(t_1) = t_n + d$		
$f(t_4) = x_n + d$		
$f(t_5) = x_n + 2d$		
Value of i and j	$f(t_{i+2})$	$f(x_j)$
$0 \le i \le 1$	$s + id$	$-$
$1 \le j \le n$	$-$	$s + (j+1)d$

Table 3.10 Labeling of Edges for the tree T_1^4

Value of i and j	$g(t_i t_{i+1})$	$g(t_2 x_j)$
$1 \le j \le 4$	$2s + 2(q-1)d - (f(t_i) + f(t_{i+1}))$	$-$
$1 \le j \le n$	$-$	$2s + 2(q-1)d - (f(t_2) + f(x_j))$

From the above table we find that f *and* g are injective and $f(t_i) + f(t_{i+1}) + g(t_i t_{i+1})$ and $f(t_2) + f(x_j) + g(t_2 x_j)$ are constant equal to 2(s+(q−1)d) .Hence we concluded that the tree T_1^4 admits (S,d) magic labeling.

Example 3.3.2 The (S,d) magic labeling of T_1^4 is shown below

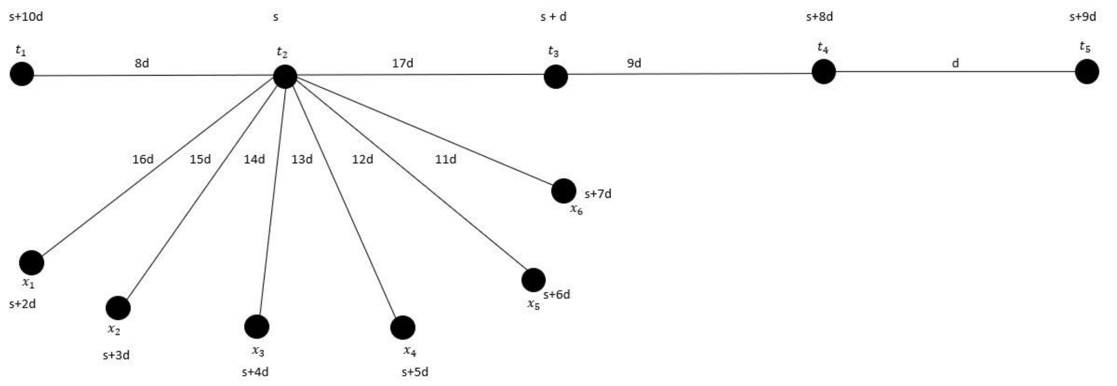

Figure 3.5 Tree of diameter 4, T_1^4

Theorem 3.3.3 The tree T_2^4 obtained by attaching n pendant edges to the second internal vertex of the path P_5 is a (S,d) magic graph.

Proof: Let $V(T_2^4) = \{t_i : 1 \leq i \leq 5\} \cup \{x_j : 1 \leq j \leq n\}$ and

$E(T_2^4) = \{t_i t_{i+1} : 1 \leq i \leq 4\} \cup \{t_2 x_j;\ 1 \leq j \leq n\}$

$p = n + 5$ and $q = n + 4$

Define $f: V(G) \to \{s, s+d, s+2d, \ldots . s+(q+1)d\}$ to label the vertices as follows

$f(t_1) = x_n$+2d

$f(t_2) = s$

$f(t_3) = s + d$

$f(t_4) = s$+2d

$f(t_5) = x_n$+d

$f(x_j) = s + (j+2)d; 1 \leq j \leq n$

Define $g: E(G) \to \{d, 2d, 3d, \ldots .2(q-1)d\}$ to label the edges as follows

$g(t_i t_{i+1}) = 2s + 2(q-1)d - (f(t_i) + f(t_{i+1}): 1 \leq i \leq 4$

$g(t_3 x_j) = 2s + 2(q-1)d - (f(t_3) + f(x_j): 1 \leq j \leq n$

Table 3.11 Labeling of Vertices for the tree T_2^4

$f(t_1) = x_n + 2d$		
$f(t_2) = s$		
$f(t_5) = x_n + d$		
Value of i and j	$f(t_{i+2})$	$f(x_j)$
$1 \leq i \leq 2$	$s + id$	$-$
$1 \leq j \leq n$	$-$	$s + (j+2)d$

Table 3.12 Labeling of Edges for the tree T_2^4

Value of i and j	$g(t_i t_{i+1})$	$g(t_3 x_j)$
$1 \leq j \leq 4$	$2s + 2(q-1)d - (f(t_i) + f(t_{i+1}))$	$-$
$1 \leq j \leq n$	$-$	$2s + 2(q-1)d - (f(t_3) + f(x_j))$

From the above table we find that f *and* g are injective and $f(t_i) + f(t_{i+1}) + g(t_i t_{i+1})$ and $f(t_3) + f(x_j) + g(t_3 x_j)$ are constant equal to 2(s+(q−1)d) .Hence we concluded that the tree T_2^4 admits (S,d) magic labeling.

Example 3.3.4 The (S,d) magic labeling of T_2^4 is shown below

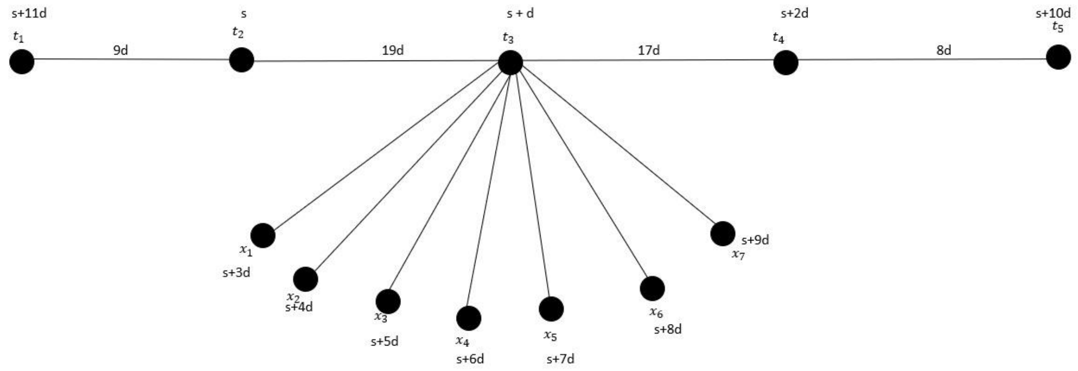

Figure 3.6 Tree of diameter 4, T_2^4

Theorem 3.3.5 The tree T_4^4 obtained by attaching m,n pendant edges to the first and second internal vertex of the path P_5 is a (S,d) magic graph.

Proof: Let $V(T_4^4) = \{t_i : 1 \leq i \leq 5\} \cup \{x_j : 1 \leq j \leq n\} \cup \{y_k : 1 \leq k \leq m\}$ and

$E(T_4^4) = \{t_i t_{i+1} : 1 \leq i \leq 4\} \cup \{t_2 x_j ; 1 \leq j \leq n\} \cup \{t_3 y_k : 1 \leq k \leq m\}$

$p = m + n + 5$ and $q = m + n + 4$

Define $f: V(G) \rightarrow \{s, s + d, s + 2d, \ldots . s + (q + 1)d\}$ to label the vertices as follows

$f(t_i) = x_n + id;\ \ 1 \leq i \leq n - 1$

$f(t_n) = y_m + d$

$f(y_k) = t_{n-1} + kd;\ 1 \leq k \leq m$

$f(x_{j+1}) = s + jd;\ 0 \leq j \leq n - 1$

Define $g: E(G) \rightarrow \{d, 2d, 3d, \ldots .2(q - 1)d\}$ to label the edges as follows

$g(t_i t_{i+1}) = 2s + 2(q - 1)d - (f(t_i) + f(t_{i+1}) : 1 \leq i \leq 4$

$g(t_2 x_j) = 2s + 2(q - 1)d - (f(t_2) + f(x_j) : 1 \leq j \leq n$

$g(t_3 y_k) = 2s + 2(q - 1)d - (f(t_3) + f(y_k) : 1 \leq k \leq m$

Table 3.13 Labeling of Vertices for the tree T_4^4

$f(t_n) = y_m + d$			
Value of i, j and k	$f(t_i)$	$f(y_k)$	$f(x_{j+1})$
$1 \leq i \leq n - 1$	$x_n + id$	$-$	$-$
$1 \leq k \leq m$	$-$	$t_{n-1} + kd$	$-$
$1 \leq j \leq n$	$-$	$-$	$s + jd$

Table 3.14 Labeling of Edges for the tree T_4^4

Value of i, j and k	$g(t_i t_{i+1})$	$g(t_2 x_j)$	$g(t_3 y_k)$
$1 \le j \le 4$	$2s + 2(q-1)d - (f(t_i) + f(t_{i+1}))$	$-$	$-$
$1 \le j \le n$	$-$	$2s + 2(q-1)d - (f(t_2) + f(x_j))$	$-$
$1 \le k \le m$	$-$	$-$	$2s + 2(q-1)d - (f(t_3) + f(y_k))$

From the above table we find that f and g are injective and $f(t_i) + f(t_{i+1}) + g(t_i t_{i+1})$, $f(t_2) + f(x_j) + g(t_2 x_j)$ and $f(t_3) + f(y_k) + g(t_3 y_k)$ are constant equal to 2(s+(q−1)d) .Hence we concluded that the tree T_4^4 admits (S,d) magic labeling.

Example 3.3.6 The (S, d) magic labeling of T_4^4 is shown below

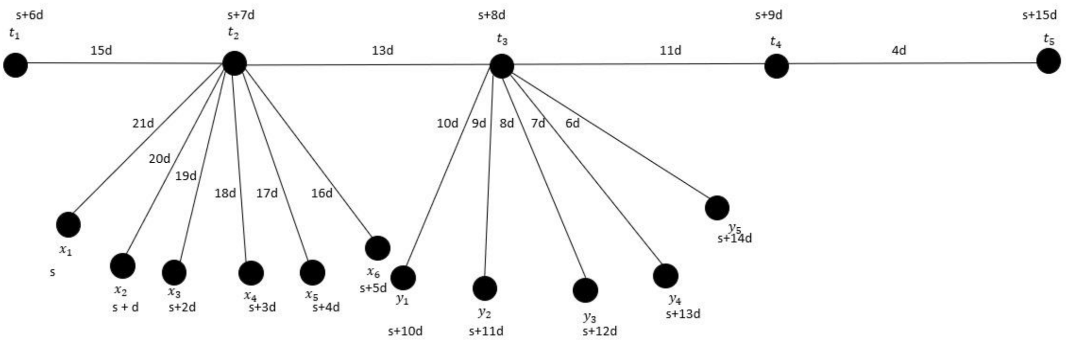

Figure 3.7 Tree of diameter 4, T_4^4

IV.CONCLUSIONS:

In this paper, we showed that some trees with a diameter less than or equal to four can be labeled as (S, d) -magic graphs. This method can also be extended to other types of graphs, enabling further exploration of (S, d) -magic labeling across various graph families.

V. REFERENCES:

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