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# Fixed Point Results And Stability In Fuzzy Metric Space

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***Abstract:*** The stability theorem, together with the existence and uniqueness of fixed points for fuzzy metric space are presented in this study. Using the notation of expansive map.

***Keyword* -** Fixed Point (FP), Fuzzy Metric Space (FMS), Stability.

## I. Introduction

FMS has been introduced in several ways by Kaleva and Seikkala [5], and Karmosil and Michalek [6]. The FP theorem of FMS was improved upon by Grabiec [4] in 1988. Almost all theoretical and applied mathematics disciplines have advanced fuzzy concepts. The concept of Banach contraction mapping was extended to probabilistic metric spaces by Sehgal et al.[8]. The field of FP theory has experienced substantial growth, resulting in numerous extensions and generalization of the findings of Sehgal et al. [8]. In dynamical systems, the concept of stability is related to limiting behaviours. In dynamical systems, stability is understood in various ways [7]. There are several notions of stability in Here, the fuzzy hausdorff metric is used to describe fuzzy stability for fixed point sets in FMS. The definition for the stability of FP sets in metric spaces is imprecise [2]and can be extended in this way. These metric space difficulties have been discussed in a number of works. We proved a stability theorem for fuzzy metric spaces and looked into the existence and uniqueness of FP for expanded map.

## II. PRELIMINARIES

**Definition 2.1** [5] Let (Â, Ê,♦) is said to be fuzzy metric space if Â is an arbitrary set, ♦ is continuous ŧ -norm and Ê is a fuzzy set on $Â^2$ x [0,∞) satisfying the following conditions;

(V1) Ê(ɱ,η,ŧ) = 0,
(V2) Ê(ɱ,η,ŧ) = 1 iff ɱ=η and ŧ > 0 ,
(V3) Ê(ɱ,η,ŧ) = Ê(η,ɱ,ŧ),
(V4) Ê(ɱ,η,ŧ) ♦ Ê(η,õ,p)≤ Ê(ɱ,õ,p+ ŧ),
(V5) Ê(ɱ,η, ♦) : [0,∞) → [0,1] is continuous, ɱ,η,õ ∈ Â and p,ŧ > 0 ,

**Definition 2.2** [5] A mapping [0,1] x [0,1] →[0,1] is said to be ŧ -norm if following hold:

(V1) ♦ is commutative as well as associative,
(V2) 1 ♦ c1 = c1 whenever c1 ∈ [0; 1],
(V3) c1 ♦ c2 ♦ c3 ♦ c4 whenever c1 ≤ c3 and c2 ≤ c4, for each c1; c2; c3; c4 ∈ [0; 1],
The t-norm ♦ is called continuous if it is continuous as a mapping.

**Definition 2.3** Let (Â, Ê,♦) be a FMS, A point ꝑ ∈ Â is called a fixed point of a mapping Û: Â → Â if Û(ꝑ)= ꝑ.

**Definition 2.4** Let (Â, Ê,♦) be a FMS. ŝ: Â → Â is said to be expensive map if there exits ά,έ,ή ≥ 0,

With ά+ή ≤ 1 such that

Ê[ŝ (ꝑ), ŝ (ƚ), ŧ] ≤ άÊ (ꝑ, ƚ, ŧ) + έ Ê[ꝑ, ŝ (ƚ), ŧ] + ,ή Ê[ƚ, ŝ (ƚ), ŧ] ………(1.1)

**Definition 2.5** Let (Â, Ê,♦) be a FMS. Let ŝ,Û be self mappings in FMS. Û is said to be a ŝ expensive map if there exits ά,έ,ή ≥ 0, Ĝ with ά+ή ≤ 1 such that

Ê[Û(ŝ (ꝑ)), Û(ŝ (ƚ)), ŧ] ≤ άÊ [Û (ꝑ), Û (ƚ), ŧ] + έ Ê[Û(ꝑ), Û(ŝ(ƚ)), ŧ] + ή Ê[ŝ(ƚ)), Û(ŝ(ƚ)), ŧ] ….(1.2)

**Remark** If Û=1, is the identity function , (1.1) we obtain Û is expensive map.

**Definition 2.6** Let (Â, Ê,♦) be a FMS, {$ꝑ_n$} be a sequence in Â and ꝑ ∈ Â such that

(i) {$ꝑ_n$} is said to be a convergent sequence if $\lim_{n\to\infty}$ Ê($ꝑ_n$, $ꝑ_m$) = Ê(ꝑ, ꝑ) .

(ii) {$ꝑ_n$} is said to be a cauchy sequence if $\lim_{n,m\to\infty}$ Ê($ꝑ_n$, $ꝑ_m$) is exits and finite.

(iii) If every cauchy sequence {$ꝑ_n$} in Â, there exits ꝑ ∈ Â such that $\lim_{n,m\to\infty}$ Ê($ꝑ_n$, $ꝑ_m$) = Ê(ꝑ, ꝑ) = $\lim_{n\to\infty}$ Ê($ꝑ_n$, ꝑ).

**Lemma 2.7** Let (Â, Ê,♦) be a FMS. Û be a self mapping in FMS. If Û is continuous at ꝑ ∈ Â, then for every sequence {$ꝑ_n$} in Â such that {$ꝑ_n$} →ν, we have Û {$ꝑ_n$} → Û {ν} that is

$\lim_{n\to\infty}$ Ê (Û($ꝑ_n$), Û(ν)) = Ê(Û (ν), Û (ν)).

**Definition 2.8** Let (Â, Ê,♦) be a FMS. Ŝ, Û be a self mapping in FMS. ꝑ∗ be the FP of Ŝ. Let { Û ($ꝑ_n$)} be sequence that is

Û ($ꝑ_{n+1}$) = Ĝ(Ŝ, Û, $ꝑ_n$, ŧ), n ∈ N ……………(1.3)

Where $ꝑ_0$ is the initial point in Â and Ĝ is function. Let {Û ($ꝑ_n$)} converges to Û(ꝑ∗).

Let {Û ($ƚ_n$)} be a arbitrary sequence in Â, and $ß_n$ = Ê (Û($ƚ_{n+1}$) ,Ĝ(Ŝ, Û, $ƚ_n$, ŧ), n ∈ N is said to be stable point iff $\lim_{n\to\infty} ß_n = 0$ => $\lim_{n\to\infty}$ Û($ƚ_n$) = Û (ꝑ ∗).

**Definition 2.9** Let (Â, Ê,♦) be a FMS. Ŝ, Û be a self mapping in FMS and $ꝑ_0$ is the initial point in Â, sequence {Û ($ꝑ_n$)} ⊂ Â defined by {Û($ꝑ_{n+1}$)} = {Û(Ŝ ($ꝑ_n$))} = {Û($Ŝ^n$ ($ꝑ_0$))} is said to be iteration. ……………….(1.4)

**Lemma 2.10** Let ($ꝑ_n$), ($ƚ_n$) be a sequence of positive numbers and $0 \le \varphi < 1$. So that

$ƚ_{n+1} \le \varphi$ ($ƚ_n$ + $ꝑ_n$) ∀ n ∈ N. If $\lim_{n\to\infty} ꝑ_n = 0$ then $\lim_{n\to\infty} ƚ_n = 0$ .

## III.MAIN RESULTS

**Theorem 3.1** Let (Â, Ê,♦) be a FMS and Û : Â → Â be a continuous and convergent mapping. If Ŝ: Â → Â is a expansive map such that ά+ή ≤ 1 , and ά + έ < 1then Ŝ has a unique FP. The sequence {Û (ꝑ$_n$)} convergence to Û(ꝑ∗), where ꝑ∗ is FP of Ŝ .

**Proof :** Let ꝑ$_0$ is the initial point in Â , now we define

$$Ê\left[Û(ꝑ_{n+1}), Û(ꝑ_n), ŧ\right] = Ê\left[Û(Ŝ(ꝑ_n)), Û(Ŝ(ꝑ_{n-1})), ŧ\right]$$
$$\leq ά\, Ê\{[Û(ꝑ_n), Û(ꝑ_{n-1}), ŧ]\} + έÊ\{[Û(ꝑ_n), Û(Ŝ(ꝑ_{n-1})), ŧ]\} + ήÊ\{[Û(ꝑ_{n-1}), Û(Ŝ(ꝑ_{n-1})), ŧ]\}$$
$$= ά\, Ê\{[Û(ꝑ_n), Û(ꝑ_{n-1}), ŧ]\} + έÊ\{[Û(ꝑ_n), Û(ꝑ_n), ŧ]\} + ήÊ\{[Û(ꝑ_{n-1}), Û(ꝑ_n), ŧ]\}$$
$$= ά\, Ê\{[Û(ꝑ_n), Û(ꝑ_{n-1}), ŧ]\} + έ.1 + ήÊ\{[Û(ꝑ_{n-1}), Û(ꝑ_n), ŧ]\}$$

By the definition {} and{}

$$= ά\, Ê\{[Û(ꝑ_n), Û(ꝑ_{n-1}), ŧ]\} + ήÊ\{[Û(ꝑ_{n-1}), Û(ꝑ_n), ŧ]\}$$
$$= (ά + ή)\{Ê[Û(ꝑ_n), Û(ꝑ_{n-1}), ŧ]\} \quad \text{......................(A)}$$

Also we obtain

$$Ê\left[Û(ꝑ_n), Û(ꝑ_{n-1}), ŧ\right] = Ê\left[Û(Ŝ(ꝑ_{n-1})), Û(Ŝ(ꝑ_{n-2})), ŧ\right]$$
$$\leq ά\, Ê\{[Û(ꝑ_{n-1})\ Û(ꝑ_{n-2}), ŧ]\} + έÊ\{[Û(ꝑ_{n-1}), Û(Ŝ(ꝑ_{n-2})), ŧ]\} + ήÊ\{[Û(ꝑ_{n-2}), Û(Ŝ(ꝑ_{n-2})), ŧ]\}$$
$$= ά\, Ê\{[Û(ꝑ_{n-1})\ Û(ꝑ_{n-2}), ŧ]\} + έÊ\{[Û(ꝑ_{n-1}), Û(ꝑ_{n-1}), ŧ]\} + ήÊ\{[Û(ꝑ_{n-2}), Û(ꝑ_{n-1}), ŧ]\}$$
$$= ά\, Ê\{[Û(ꝑ_{n-1})\ Û(ꝑ_{n-2}), ŧ]\} + έ.1 + ήÊ\{[Û(ꝑ_{n-2}), Û(Ŝ(ꝑ_{n-2})), ŧ]\}$$

By the definition {2.4} and{2.5}

$$= (ά + ή)\{Ê[Û(ꝑ_{n-1}), Û(ꝑ_{n-2}), ŧ]\} \quad \text{..............................(B)}$$

Now from equation A and B, We have

$$Ê\left[Û(ꝑ_{n+1}), Û(ꝑ_n), ŧ\right] \leq (ά + ή)^2\{Ê[Û(ꝑ_{n-1}), Û(ꝑ_{n-2}), ŧ]\}$$

we get

$$Ê\left[Û(ꝑ_{n+1}), Û(ꝑ_n), ŧ\right] \leq (ά + ή)^{n-1}\{Ê[Û(ꝑ_{n-1}), Û(ꝑ_{n-2}), ŧ]\} \quad \text{.......................(C)}$$

For n > m, we have

$$Ê\left[Û(ꝑ_m), Û(ꝑ_n), ŧ\right] \leq \{(ά + ή)^{m-1} + (ά + ή)^m + (ά + ή)^{m+1} + .....+ (ά + ή)^{n-2}\}\ Ê\left[Û(ꝑ_2), Û(ꝑ_1), ŧ\right] \text{...............(D)}$$

It follows that { Û (ꝑ$_n$)} be a Cauchy sequence with $\lim_{n,m\to\infty} Ê[Û(ꝑ_n), Û(ꝑ_m)] = 0$ and (Â, Ê,♦) be a Complete FMS, there exits ꝑ$_0$ in Â such that

$$\lim_{n,m\to\infty} Ê[Û(ꝑ_n), Û(ꝑ_m),, ŧ] = Ê\left[ꝑ_0, ꝑ_0, ŧ\right] = 0 \quad \text{...............(E)}$$

The mapping Û is convergent and sequence $Û(ꝑ_n)$ is convergent to Û (ꝑ∗), then the *sequence* $\{ꝑ_n\}$ is convergent to {ꝑ∗} , so there exits to ꝑ∗ in Â such that

$$\lim_{n,m\to\infty} Ê[Û(ꝑ_n), Û(ꝑ_m), ŧ] = Ê[ꝑ *, ꝑ *, ŧ] \quad \text{...............(F)}$$

Because Ŝ is continuous, by Lemma (), we have

$$\lim_{n\to\infty} Ê[Û(ꝑ_n), Û(ꝑ *)), ŧ] = Ê\left[Û(ꝑ *), Û(ꝑ *), ŧ\right] \text{................(G)}$$

From equation (E) and (F), we obtain $ꝑ_0 = \hat{S}(ꝑ*)$.

$$\hat{E}\left[\hat{U}(\hat{S}(ꝑ*)), \hat{U}(ꝑ*), ŧ\right] \leq \hat{E}\left[\hat{U}(\hat{S}(ꝑ*)), \hat{U}(\hat{S}(ꝑ_n)), ŧ\right] + \hat{E}\left[\hat{U}(\hat{S}(ꝑ_n)), \hat{U}(ꝑ*), ŧ\right]$$
$$\leq \acute{\alpha}\, \hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_n), ŧ]\} + \acute{\varepsilon}\hat{E}\{[\hat{U}(ꝑ*), \hat{U}(\hat{S}(ꝑ_n)), ŧ]\} + \acute{\eta}\hat{E}\{[\hat{U}(ꝑ_n), \hat{U}(\hat{S}(ꝑ_n)), ŧ]\} + \hat{E}\left[\hat{U}(\hat{S}(ꝑ_n)), \hat{U}(ꝑ*), ŧ\right]$$
$$\leq \acute{\alpha}\, \hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_n), ŧ]\} + \acute{\varepsilon}\hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_{n-1}), ŧ]\} + \acute{\eta}\hat{E}\{[\hat{U}(ꝑ_n), \hat{U}(ꝑ_{n-1}), ŧ]\} + \hat{E}\left[\hat{U}(ꝑ_{n-1}), \hat{U}(ꝑ*), ŧ\right]$$
$$\leq \acute{\alpha}\, \hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_n), ŧ]\} + \acute{\eta}\hat{E}\{[\hat{U}(ꝑ_n), \hat{U}(ꝑ_{n-1}), ŧ]\} + (\acute{\varepsilon}+1)\hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_{n-1}), ŧ]\}$$
$$\leq \acute{\alpha}\, \hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_n), ŧ]\} + \acute{\eta}\hat{E}\{[\hat{U}(ꝑ_n), \hat{U}(ꝑ*), ŧ,] + \hat{U}(ꝑ*), \hat{U}(ꝑ_{n-1}), ŧ]\} + (\acute{\varepsilon}+1)\hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_{n-1}), ŧ]\}$$
$$\leq (\acute{\alpha}+\acute{\eta})\, \hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_n), ŧ]\} + (\acute{\varepsilon}+\acute{\eta}+1)\hat{E}\{[\hat{U}(ꝑ*), \hat{U}(ꝑ_{n-1}), ŧ]\}$$
$$= 0 \text{ as } n \to \infty$$

$\hat{E}\left[\hat{U}(\hat{S}(ꝑ*)), \hat{U}(ꝑ*), ŧ\right] = 0. \Rightarrow \hat{U}(\hat{S}(ꝑ*)) = \hat{U}(ꝑ*)$. *Since* $\hat{U}$ is one to one,

we have $\hat{S}(ꝑ*) = ꝑ*$. Hence $ꝑ*$ is a FP of $\hat{S}$.

To show that $ꝑ*$ is a unique FP of $\hat{S}$. Let another FP $ƀ^*$ such that $ꝑ* \neq ƀ^*$.

That is $\hat{U}(ꝑ*) = ꝑ*$ and $\hat{U}(ƀ^*) = ƀ^*$. We get

$$\hat{E}\left[\hat{U}(ƀ^*), \hat{U}(ꝑ*)), ŧ\right] = \hat{E}\left[\hat{U}(\hat{S}(ƀ^*)), \hat{U}(\hat{S}(ꝑ*)), ŧ\right]$$
$$\leq \acute{\alpha}\, \hat{E}\{[\hat{U}(ƀ^*), \hat{U}(ꝑ*)), ŧ]\} + \acute{\varepsilon}\hat{E}\{[\hat{U}(ƀ^*), \hat{U}(\hat{S}(ꝑ*)), ŧ]\} + \acute{\eta}\hat{E}\{[\hat{U}(ꝑ*), \hat{U}(\hat{S}(ꝑ*)), ŧ]\}$$
$$= (\acute{\alpha}+\acute{\varepsilon})\hat{E}\{[\hat{U}(ƀ^*), \hat{U}(ꝑ*), ŧ]\}$$

$$1-(\acute{\alpha}+\acute{\varepsilon})\hat{E}\{[\hat{U}(ƀ^*), \hat{U}(ꝑ*), ŧ]\} = 0$$

We have $(\acute{\alpha}+\acute{\varepsilon}) < 1$, then $\hat{E}\{[\hat{U}(ƀ^*), \hat{U}(ꝑ*), ŧ]\} = 0$ then $\hat{U}(ƀ^*) = \hat{U}(ꝑ*)$.

We get $ƀ^* = ꝑ*$.

**Theorem 3.2** Under the hypotheses of theorem 1, the sequence $\hat{U}(ƅ_n)$ is stable.

**Proof :** Suppose that $\hat{U}(ƅ_n)$ is an arbitrary sequence in $\hat{A}$ and $ß_n = \hat{E}\{[\hat{U}(ƅ_{n+1}), \hat{U}(\hat{S}(ƅ_n)), ŧ]\}$.

Let $\lim_{n\to\infty} ß_n = 0$. We obtain

$$\hat{E}\{[\hat{U}(ƅ_{n+1}), \hat{U}(ꝑ*), ŧ]\} \leq \hat{E}\{[\hat{U}(ƅ_{n+1}), \hat{U}(\hat{S}(ƅ_n)), ŧ]\} + \hat{E}\{[\hat{U}(\hat{S}(ƅ_n)), \hat{U}(ꝑ*), ŧ]\}$$
$$= \hat{E}\{[\hat{U}(ƅ_{n+1}), \hat{U}(\hat{S}(ƅ_n)), ŧ]\} + \hat{E}\{[\hat{U}(\hat{S}(ƅ_n)), \hat{U}(\hat{S}(ꝑ*)), ŧ]\}$$

$$\leq ß_n + (ά + ή)\ \hat{E}\{[\ \hat{U}(ƀ_n), \hat{U}(ꝥ *), ŧ\ ]\}$$

$$\lim_{n\to\infty} ß_n = 0 \text{ by lemma } \lim_{n\to\infty} \hat{U}(ƀ_n) = \hat{U}\ (ꝥ *).$$

Conversely, $\lim_{n\to\infty} \hat{E}\{[\ \hat{U}(ƀ_{n+1}), \hat{U}(ꝥ *), ŧ\ ]\} = 0$, then by lemma 2, we have

$$ß_n = \hat{E}\{[\ \hat{U}(ƀ_{n+1}), \hat{U}(\hat{S}(ƀ_n)), ŧ\ ]\}$$

$$\leq \hat{E}\{[\ \hat{U}(ƀ_{n+1}), \hat{U}(ꝥ *), ŧ\ ]\} + \hat{E}\{[\ \hat{U}(ꝥ *), \hat{U}(\hat{S}(ƀ_n)),\ ŧ\ ]\}$$

$$= \hat{E}\{[\ \hat{U}(ƀ_{n+1}), \hat{U}(ꝥ *), ŧ\ ]\} + \hat{E}\{[\ \hat{U}(\hat{S}(ƀ_n)), \hat{U}(\hat{S}(ꝥ *)), ŧ\ ]\}$$

$$= \hat{E}\{[\ \hat{U}(ƀ_{n+1}), \hat{U}(ꝥ *), ŧ\ ]\} + (ά + ή)\ \hat{E}\{[\ \hat{U}(ƀ_n), \hat{U}(ꝥ *), ŧ\ ]\}$$

$$= 0 \text{ as } n \to \infty$$

Therefore $\hat{U}(ƀ_n)$ is stable.

## IV.CONCLUSIONS –

Our theorem is proved using the t-norm assumption. We have given the stability result for FMS and established the existence and uniqueness of the FP of expanded map. It is also observed that sequences of mappings take place when stability is granted. In the case of mappings, we can acquire several fixed points.

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