IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

STABILITY OF FOURTH ORDER PARTIAL DIFFERENTIAL EQUATION

V. P. Sonalkar

Asst. Prof.
Department of Mathematics, S. P. K. Mahavidyalaya Sawantwadi,
Maharashtra- 416510, India

***Abstract*:** In this paper, we prove the Hyers-Ulam-Rassias (HUR) stability of fourth order partial differential equation:

$$p(x,t)u_{xxxx}(x,t) + p_x(x,t)\,u_{xxx}(x,t) + p(x,t)\,u_{xx}(x,t) + p_x(x,t)\,u_x(x,t) = g(x,t,u(x,t)).$$

Index Terms - Hyers-Ulam-Rassias Stability, Banach's contraction principle, partial differentialequation, Functional equations.

I. INTRODUCTION

In 1940, S. M. Ulam [15], gave a well-known talk on stability problems for several functional equations. In the talk, Ulam discussed a problem concerning the stability of group homomorphism. In 1941, D. H. Hyers [5] gave a partial solution to Ulam's problem. There have been of publications on stability of solutions to differential equations [3, 6, 7] and partial differential equations [8, 9]. This stability is now referred to as the Hyers Ulam (HU) stability and its various extensions has been named with additional word. Hyers Ulam Rassias (HUR) stability is one such extension. In [10] and [11], HUR stability for linear differential operators of n^{th} order with non-constant coefficients is invested. HUR stability for special types of non-linear equations have been studied in [1, 2, 12, 13]. In 2011, Gordji et al. [4], established the HUR stability of non-linear partial differential equations by using Banach's Contraction Principle. In 2019, Sonalkar et. al. [14], proved the HUR Stability of linear partial differential equations by using Laplace transform method. In this paper, by using the result of [4], we prove the HUR stability of fourth order partial differential equation:

$$p(x,t)u_{xxxx}(x,t) + p_x(x,t)u_{xxx}(x,t) + p(x,t)\,u_{xx}(x,t) + p_x(x,t)\,u_x(x,t) = g(x,t,u(x,t)). \quad (1.1)$$

Here $p: J\times J \to R^+$ be a differentiable function at least once w. r. t. both the arguments and $p(x,t) \neq 0,\ \forall x, t \in J$, $J = [a, b]$ be a closed interval and $g: J \times J \times R \to R$ be a continuous function.

Definition 1.1: A function $u: J\times J \to R$ is called a solution of equation (1.1) if $u \in C^4(J\times J)$ and satisfies the equation (1.1).

II. PRELIMINARIES

Definition 2.1: The equation (1.1) is said to be HUR stable if the following holds:

Let $\varphi: J \times J \to (0, \infty)$ be a continuous function. Then $\exists$ a continuous function $\Psi: J \times J \to (0, \infty)$, which depends on φ such that whenever $u: J \times J \to R$ is a continuous function with

$$|p(x,t)u_{xxxx}(x,t) + p_x(x,t)u_{xxx}(x,t) + p(x,t)\,u_{xx}(x,t) + p_x(x,t)u_x(x,t) - g(x,t,u(x,t))| \le \varphi(x,t), \quad (2.1)$$

there exists a solution $u_0 : J \times J \to R$ of (1.1) such that

$$|u(x,t) - u_0(x,t)| \le \Psi(x,t), \qquad \forall\, (x,t) \in J \times J.$$

We need the following result.

Banach Contraction Principle:

Let (Y, d) be a complete metric space, then each contraction map $T: Y \to Y$ has a unique fixed point, that is, there exists $b \in Y$ such that $Tb = b$. Moreover, $d(b,w) \leq \frac{1}{(1-\alpha)} d(w,Tw)$, $\forall\, w \in Y$ and $0 \leq \alpha < 1$.

Using the results from Gordji et al. [4], we establish the following result.

III. MAIN RESULT

In this section we prove HUR stability of fourth order partial differential equation (1.1).

Theorem 3.1: Let $c \in J$. Let p and g be as in (1.1) with additional conditions:

(i) $p(x,t) \geq 1, \ \forall\, x, t \in J$.

(ii) $\varphi: J \times J \to (0,\infty)$ be a continuous function and $M: J \times J \to [1,\infty)$ be an integrable function.

(iii) Assume that there exists γ, $0 < \gamma < 1$ such that

$$\int_c^x M(\tau,t)\varphi(\tau,t)d\tau \ \leq \gamma\varphi(x,t), \tag{3.1}$$

$$\int_c^x \int_c^\tau \int_c^\alpha M(\beta,t)\varphi(\beta,t)d\beta d\alpha d\tau \ \leq \gamma\varphi(x,t) \tag{3.2}$$

and

$$K(x,t,u(x,t)) \ = p(x,t)^{-1}\left[p(c,t)u_{xxx}(c,t) - p(x,t)u_x(x,t) + p(c,t)u_x(c,t) + \int_c^x g\big(\tau,t,u(\tau,t)\big)\, d\tau\right]. \tag{3.3}$$

Suppose that the following holds:

C1: $|K(\tau, t, l(\tau, t)) - K(\tau, t, m(\tau, t))| \leq M(\tau, t)|l(\tau, t) - m(\tau, t)|, \forall\, \tau, t \in J$ and $l, m \in C(J \times J)$.

C2 : $u : J \times J \to \mathbb{R}$ be a function satisfying the inequality (2.1).

Then there exists a unique solution $u_0 : J \times J \to \mathbb{R}$ of the equation (1.1) of the form

$$u_0(x,t) = u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u_0(\beta,t))d\beta d\alpha d\tau$$

such that

$$|u(x,t) - u_0(x,t)| \leq \frac{\gamma}{(1-\gamma)}\varphi(x,t), \ \forall\, x,t \in J.$$

Proof: Consider

$$| p(x,t)u_{xxxx}(x,t) + p_x(x,t)\, u_{xxx}(x,t) + p(x,t)\, u_{xx}(x,t) + p_x(x,t)\, u_x(x,t) - g(x,t,u(x,t))|$$

$$= |\{p(x,t)u_{xxx}(x,t)\}_x + \{p(x,t)u_x(x,t)\}_x - g(x,t,u(x,t))|.$$

From the inequality (2.1), we get

$$|\{p(x,t)u_{xxx}(x,t)\}_x + \{p(x,t)u_x(x,t)\}_x - g(x,t,u(x,t))| \leq \varphi(x,t).$$

$$\Rightarrow -\varphi(x,t) \leq \{p(x,t)u_{xxx}(x,t)\}_x + \{p(x,t)u_x(x,t)\}_x - g(x,t,u(x,t)) \leq \varphi(x,t). \tag{3.4}$$

$$\Rightarrow \{p(x,t)u_{xxx}(x,t)\}_x + \{p(x,t)u_x(x,t)\}_x - g(x,t,u(x,t)) \leq \varphi(x,t).$$

Integrating from c to x we get,

$$p(x,t)u_{xxx}(x,t) - p(c,t)u_{xxx}(c,t) + p(x,t)u_x(x,t) - p(c,t)u_x(c,t) - \int_c^x g\big(\tau,t,u(\tau,t)\big)\, d\tau \leq \int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow p(x,t)\left\{u_{xxx}(x,t) - p(x,t)^{-1}\left[p(c,t)u_{xxx}(c,t) - p(x,t)u_x(x,t) + p(c,t)u_x(c,t) + \int_c^x g\big(\tau,t,u(\tau,t)\big)\, d\tau\right]\right\}$$
$$\leq \int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow \left\{u_{xxx}(x,t) - p(x,t)^{-1}\left[p(c,t)u_{xxx}(c,t) - p(x,t)u_x(x,t) + p(c,t)u_x(c,t) + \int_c^x g\big(\tau,t,u(\tau,t)\big)\, d\tau\right]\right\}$$
$$\leq p(x,t)^{-1}\int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow \left\{u_{xxx}(x,t) - p(x,t)^{-1}\left[p(c,t)u_{xxx}(c,t) - p(x,t)u_x(x,t) + p(c,t)u_x(c,t) + \int_c^x g\big(\tau,t,u(\tau,t)\big)\, d\tau\right]\right\} \leq \int_c^x \varphi(\tau,t)d\tau,$$

$(\because p(x,t) \geq 1)$.

$$\Rightarrow \{u_{xxx}(x,t) - K(x,t,u(x,t))\} \leq \int_c^x \varphi(\tau,t)d\tau.$$

where $K(x,t,u(x,t))$ is given by equation (3.3).

Since $M: J \times J \to [1,\infty)$ be an integrable function, we have

$$\{u_{xxx}(x,t) - K(x,t,u(x,t))\} \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$\{u_{xxx}(x,t) - K(x,t,u(x,t))\} \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \leq \gamma\varphi(x,t).$$

$$\Rightarrow \{u_{xxx}(x,t) - K(x,t,u(x,t))\} \leq \varphi(x,t), \quad (\because 0 < \gamma < 1). \tag{3.5}$$

Again, integrating from c to x we get,

$$u_{xx}(x,t) - u_{xx}(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \int_c^x \varphi(\tau,t)d\tau.$$

Since $M: J \times J \to [1,\infty)$ be an integrable function, we have

$$u_{xx}(x,t) - u_{xx}(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$u_{xx}(x,t) - u_{xx}(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \leq \gamma\varphi(x,t).$$

$$\Rightarrow u_{xx}(x,t) - u_{xx}(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \gamma\varphi(x,t).$$

$$\Rightarrow u_{xx}(x,t) - u_{xx}(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \leq \varphi(x,t), \quad (\because 0 < \gamma < 1).$$

Again, integrating from c to x we get,

$$u_x(x,t) - u_x(c,t) - \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau \leq \int_c^x \varphi(\tau,t)d\tau.$$

Since $M: J \times J \to [1,\infty)$ be an integrable function, we have

$$u_x(x,t) - u_x(c,t) - \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$u_x(x,t) - u_x(c,t) - \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \leq \gamma\varphi(x,t).$$

$$\Rightarrow u_x(x,t) - u_x(c,t) - \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau \leq \varphi(x,t), \quad (\because 0 < \gamma < 1).$$

Again, integrating from c to x we get,

$$u(x,t) - u(c,t) - \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u(\beta,t))d\beta d\alpha d\tau \leq \int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u(\beta,t))d\beta d\alpha d\tau\right] \leq \int_c^x \varphi(\tau,t)d\tau.$$

Since $M: J \times J \to [1,\infty)$ be an integrable function, we have

$$u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u(\beta,t))d\beta d\alpha d\tau\right] \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u(\beta,t))d\beta d\alpha d\tau\right] \leq \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \leq \gamma\varphi(x,t).$$

$$\Rightarrow u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u(\beta,t))d\beta d\alpha d\tau\right] \leq \gamma\varphi(x,t). \tag{3.6}$$

In a similar way, from the left inequality of (3.4), we obtain

$$-\left\{u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u(\beta,t))d\beta d\alpha d\tau\right]\right\} \leq \gamma\varphi(x,t). \tag{3.7}$$

From the inequalities (3.6) and (3.7) we get,

$$\left|u(x,t) - \left\{u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,u(\beta,t))d\beta d\alpha d\tau\right\}\right| \leq \gamma\varphi(x,t). \tag{3.8}$$

Let Y be the set of all continuously differentiable functions $l: J \times J \to$ R. We define a metric d and an operator T on Y as follow: For $l, m \in Y$

$$d(l,m) = sup_{x,t\in J}\left|\frac{l(x,t)-m(x,t)}{\varphi(x,t)}\right|$$

and the operator

$$(T_m)(x,t) = u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K\big(\beta,t,m(\beta,t)\big)d\beta d\alpha d\tau. \quad (3.9)$$

Consider,

$$d(Tl,Tm) = sup_{x,t\in J}\left|\frac{Tl(x,t)-Tm(x,t)}{\varphi(x,t)}\right|.$$

$$= sup_{x,t\in J}\left\{\frac{\int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,l(\beta,t))d\beta d\alpha d\tau - \int_c^x \int_c^\tau \int_c^\alpha K(\beta,t,m(\beta,t))d\beta d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$\le sup_{x,t\in J}\left\{\frac{\int_c^x \int_c^\tau \int_c^\alpha |K(\beta,t,l(\beta,t))-K(\beta,t,m(\beta,t))|d\beta d\alpha d\tau}{\varphi(x,t)}\right\}.$$

By using condition C1 we get,

$$d(Tl, Tm) \le sup_{x,t\in J}\left\{\frac{\int_c^x \int_c^\tau \int_c^\alpha M(\beta,t)|\, l(\beta,t)-m(\beta,t)|d\beta d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$= sup_{x,t\in J}\left\{\frac{\int_c^x \int_c^\tau \int_c^\alpha M(\beta,t)\varphi(\beta,t)\frac{|\, l(\beta,t)-m(\beta,t)|}{\varphi(\beta,t)}d\beta d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$\le sup_{x,t\in J}\left\{\frac{\int_c^x \int_c^\tau \int_c^\alpha M(\beta,t)\varphi(\beta,t)\times sup_{\beta,t\in J}\frac{|\, l(\beta,t)-m(\beta,t)|}{\varphi(\beta,t)}d\beta d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$= d(l,m)\times sup_{x,t\in J}\left\{\frac{\int_c^x \int_c^\tau \int_c^\alpha M(\beta,t)\varphi(\beta,t)\, d\beta d\alpha d\tau}{\varphi(x,t)}\right\}.$$

By using inequality (3.2) we get,

$$d(Tl,Tm) \le \gamma d(l,m).$$

By using Banach contraction principle, there exists a unique $u_0 \in Y$ such that $Tu_0 = u_0$, that is

$$u(c,t) + \int_c^x \int_c^\tau \int_c^\alpha K\big(\beta,t,u_0(\beta,t)\big)d\beta d\alpha d\tau = u_0(x,t), \quad \text{(by using equation (3.9))}$$

and

$$d(u_0,u) \le \frac{1}{(1-\gamma)}d(u,Tu). \quad (3.10)$$

Now by using inequality (3.8) we get,

$$|u(x,t)-(Tu)(x,t)| \le \gamma\varphi(x,t).$$

$$\Rightarrow \frac{|u(x,t)-(Tu)(x,t)|}{\varphi(x,t)} \le \gamma\,.$$

$$\Rightarrow sup_{x,t\in J}\left\{\frac{|u(x,t)-(Tu)(x,t)|}{\varphi(x,t)}\right\} \le \gamma.$$

Thus $d(u,Tu) \le \gamma.$ (3.11)

Again,

$$d(u_0,u) = sup_{x,t\in J}\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right|.$$

From equation (3.10) we get,

$$d(u_0,u) \le \frac{1}{(1-\gamma)}d(u,Tu).$$

$$sup_{x,t\in J}\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \le \frac{1}{(1-\gamma)}d(u,Tu).$$

$$\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \le sup_{x,t\in J}\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \le \frac{1}{(1-\gamma)}d(u,Tu).$$

$$\Rightarrow \left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \le \frac{1}{(1-\gamma)}d(u,Tu).$$

From equation (3.11) we get,

$$\left|\frac{u_0(x,t)-u(x,t)}{\varphi(x,t)}\right| \le \frac{\gamma}{(1-\gamma)}.$$

$$\left|\frac{u(x,t)-u_0(x,t)}{\varphi(x,t)}\right| \le \frac{\gamma}{(1-\gamma)}.$$

$$|u(x,t) - u_0(x,t)| \leq \frac{\gamma}{(1-\gamma)}\varphi(x,t), \qquad \forall\, x,t \in J.$$

Hence the result.

IV. CONCLUSION

In this paper we have proved the HUR stability of the fourth order partial differential equation (1.1) by employing Banach's contraction principle.

REFERENCES

[1] Q. H. Alqifiary 2013, O*n Some properties of second order differential equation*, Mathematica Moravila, Vol. 17 (1), 89-94.

[2] Q. H. Alqifiary, S.M. Jung 2014. *On the Hyers -Ulam stability of differential equations of second order*, Abstract and Applied Analysis, Vol. 2014, 1-8.

[3] C. Alsina, R. Ger 1998. O*n some inequalities and stability results related to the exponential function*, J. Inequal. Appl., Vol. 2, 373–380.

[4] M. E. Gordji, Y. J. Cho, M. B. Ghaemi, B. Alizadeh 2011. *Stability of second order partial differential equations*, J. of Inequalities and Applications, Vol. 2011:81, 1-10.

[5] D. H. Hyers 1941. *On the stability of the linear functional equation*, proc. Natl., Acad. Science USA 27,222-224.

[6] A. Javadian 2015. A*pproximately n - order linear differential equations*, Inter. Jour. nonlinear analysis and applications, 6 (1), 135-139.

[7] A. Javadian, E. Sorouri, G. H. Kim, M. Eshaghi Gordji 2011. *Generalized Hyers -Ulam stability of a second order linear differential equations*, Applied Mathematics, Vol. 2011, Article ID 813137, doi: 10.1155/2011/813137.

[8] S. M. Jung 2006. *Hyers -Ulam stability of linear differential equation of first order, II*, Applied Mathematics Letters, Vol. 19, 854 - 858.

[9] S. M. Jung 2009. *Hyers -Ulam stability of linear partial differential equation of first order*, AppliedMathematics Letters, Vol. 22, 70 - 74.

[10] A. N. Mohapatra 2015. *Hyers -Ulam and Hyers - Ulam - Aoki - Rassias stability for ordinarydifferential equations*, Application and Applied Mathematics, vol.10, Issue 1,149-161.

[11] D. Popa, I. Rosa 2012. *Hyers-Ulam stability of the linear differential operator with nonconstantcoefficients*, Applied Mathematics and Computation, Vol. 212, 1562–1568.

[12] M. N. Qarawani 2012. *Hyers-Ulam stability of linear and non-linear differential equations of secondorder*, Inter. J. of Applied Mathematics Research, Vol. 1, No. 4, 422 - 432.

[13] M. N. Qarawani 2012. *Hyers-Ulam stability of a generalized second order nonlinear differential equation*, Applied Mathematics, Vol. 3,1857 - 1861.

[14] V. P. Sonalkar, A. N. Mohapatra and Y. S. Valaulikar 2019, *Hyers-Ulam-Rassias Stability of linear partial differential equation*, Journal of Applied Science and Computations, Vol. VI, Issue III, 839-846.

[15] S. M. Ulam 1960. *A collection of Mathematical problems*, Interscience publication, New York.