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HOMOTOPY INDUCING GROUP ACTIONS

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Abstract: Several mathematical concepts share a common feature that acts as a link between them. One such connection between the mathematical concepts of Group, Group Actions, and Homotopy is that they all involve projective maps. This underlying commonality allows us to establish a relationship between them, prompting a focused study of Group Actions and Homotopy to uncover this connection. While these concepts have each been the subject of extensive individual research, there has been little effort to determine if one can lead to the other and the conditions for such an interlinking function. Additionally, identifying Group Actions can be more challenging than finding Homotopic maps. Therefore, we anticipate that discovering an interlink could be beneficial. By relating these concepts on a common platform, we aim to determine if they can coexist through a single function.

As a step in this direction, this paper is a part of the work that is about the condition for functions to be Homotopic under a Group Action and conversely. Here we derive the condition for a Homotopy to induce a group action. Our aim is to find a single map which can give rise to a Homotopy and a Group Action simultaneously. The extension of Homotopy to sets, vector-fields etc., serves to meet this purpose. Similarly, we extend Group Actions to general sets (Set-Action). We obtain conditions for a Homotopy to be a group, Action. This study serves as an approach to find Group Actions using existing Homotopy. Thus, we show here that a Homotopy leads to group action between two functions with special properties and theorems on these.

Index Terms **–**Homotopy, Group Action, Induced Homotopy, Induced Group Action, Linking Group Action and Homotopy.

I. Introduction

Symmetry appears throughout nature, mathematics, art and sciences. Symmetries are expressed mathematically as groups. Symmetric transformations are expressed mathematically as the action of a group on a set. For example, reflection across some axis and rotation by 180 degrees are distinct group actions of Z2 on the plane. To qualify as a group action of a group G, a set of operations must satisfy the defining laws of G under composition [1,2]]. Groups have the power to move sets around – for instance when symmetric group Sn (set of all bijections) permutes the set {1,2, ... n}. This is possible with the help of Group Actions. These serve as useful tools when geometric objects with more symmetries are moved around. For instance, the different ways to make a bracelet having three red beads, two blue beads and two white beads. This is found to be carried out in 18 different ways which is cumbersome although possible through mere inspection [3]. The higher numbers involved, results in complex calculations which is where Group Actions play a vital role.

Continuous deformation of a function from one topological space to another constitutes a Homotopy between the two functions. We may relate the concept of Homotopy to image processing and animation. One finds many applications like facial feature tracking with several models to handle shape variations [4,5]. We would like to explore the application of Homotopy to such possibilities in future.

Both the mathematical concepts of Group Action and Homotopy are in-depth fields of study in their own respects. A natural question linking these two concepts is, how a Group Action can be transferred along a Homotopy. Several Homotopy Group Actions have been realized [4]. But the question of finding and relating these concepts to real problems or situations gave rise to the question of whether these concepts co-exist. This lead to the question whether a single function can act as a Group Action and a Homotopy concurrently [6].

The motivation for our study is the unification of mathematical concepts of Groups, Group Actions and Homotopy. We explore the possibility of unifying these concepts based on commonalities. In an attempt to unify these concepts, our study aims at finding a single function which is a Group Action and a Homotopy simultaneously. Thus, the need to generalize these concepts of Homotopy and Group Action to sets, vector-fields etc. Although unification has not been studied so far, these may exist together in real time situations. On one hand we would like to find the criteria for the existence of such a function and on the other we attempt to interpret real time problems where this unification may exist. Our study is based on this idea to explore the existence of such functions and the conditions to be met then [7].

The classical definition of Homotopy can also be interpreted as amalgamating two functions into one specifically, if f and g are mappings from $X \to Y$, then a Homotopy F is a mapping $X \times (0, 1) \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ which means F amalgamates f and g. A general set may not have a group structure. If F, a projective map from $X \times (0, 1) \to Y$ is a Homotopy between two functions 'f' and 'g': $X \to Y$, and if at all F induces a group action then it must induce a group structure on X, we work at this possibility. Throughout this paper X and Y represent sets or topological spaces. We define the usual Homotopy and extend it to define the same on sets, Vector-fields etc. [8]

Our study focusses on finding such a function and if it exists, what conditions does such a function satisfy. For this we necessarily need a function which acts as Homotopy and a Group Action simultaneously. These ideas guided the need to develop a unified theory linking the two mathematical concepts on a common platform. This paper focusses on finding the conditions to be satisfied by such a function. If successful in developing such theory, we anticipate to link these mathematical concepts to interpret real world problems [9].

Thus, the purpose of this paper is to explore the criteria a single function satisfies to be a Homotopy resulting in a Group Action. This unification idea is a completely new approach and we anticipate this to help find Homotopy leading to a Group Action. This paper contains section 1 on Introduction, section 2 deals with preliminaries, section 3 on Main Results followed by section 4 stating precisely the need of this work presented here. Section 5 is the concluding section.

Summarizing, this paper is a only a part of the work which focusses essentially on finding the condition when the existence of Homotopy leads to a Group Action.

II. Proposed Methodology

The proposed methodology for studying homotopy-inducing group actions involves a detailed analysis of the underlying algebraic and topological structures. Initially, we will define the specific group actions on topological spaces and identify the homotopy classes associated with these actions. This step includes the examination of fixed points, orbits, and the impact of group elements on the topology of the space. We will utilize tools from algebraic topology, such as homotopy groups and fundamental groups, to categorize and analyze these actions.

Subsequently, we will apply computational methods to simulate and visualize the group actions on various topological spaces. By leveraging software tools and algorithms, we can model complex spaces and observe the effects of group actions in real-time. This computational approach will help in identifying patterns and invariants that are difficult to discern analytically. Moreover, the methodology includes validating the theoretical findings with practical examples and applications, ensuring that the proposed framework is robust and applicable to real-world problems in fields like robotics, data analysis, and quantum computing.

III. PRELIMINARIES

In this section we define basic concepts of Homotopy and Group Action along with their extensions.

2.1. Definition: Homotopy

The mappings f, g: $X \to Y$ are said to be homotopic if and only if there exists a continuous map $F: X \times (0, 1) \to Y$ such that:

(i) $F(x, 0) = f(x)$ and (ii) $F(x, 1) = g(x)$, $\forall$ x. Then $f \simeq g$. [1]

We extend this Homotopy replacing the interval [0, 1] by a general set resulting in the following definition.

2.2. Definition: Set-Homotopy

Let f, g: X → Y. We say that the functions f, g are called set-homotopic or projection of sets if ∃ a mapping F: X × Z → Y where Z is some set (it could be a topological space or a group etc.,) such that F(x, a) = f(x) and F(x, b) = g(x) where 'a' and 'b' are two fixed points of Z.

This definition is silent on the continuity of f, g and F because X, Y, Z are general sets and continuity cannot be defined on these.

For applications of Homotopy to study Electrical and Magnetic fields we define what may be called vector field Homotopy.

2.3. Definition: Vector-field Homotopy

Let V and W be vector fields on E^3. A vector-Field Homotopy is defined as the mapping F: E^3 × I → T(E^3) such that:

F is continuous and F (x, 0) = v(x), F(x, 1)= w(x); where v, w: E^3→ T(E^3) i.e., v(x) and w(x) are tangents to E^3 at x.

We extend this to define Homotopy on three vector-fields.

2.4. Definition: 3-Vector field Homotopy

A 3-vector Field Homotopy is defined as the mapping F: X × I → Y where X = E^3 and Y=T(E^3) satisfying:

(1) F(x, 0) = v(x)
(2) F(x, 1/2) = w(x)
(3) F(x, 1) = z(x); where v, w, z: E^3 → T(E^3)

The converse problem of amalgamating two functions i.e., splitting a function into 2 or more functions is also important and finds many applications. In the first interpretation we seek to find a function which amalgamates three functions which will help us to find a single force combining the electrical, magnetic and gravitational fields. The converse interpretation is to find a function F which splits into three functions giving the same application as above.

For this we define a 3-mapping Homotopy:

2.5. Definition: 3-mapping Homotopy

Let X and Y be two topological spaces. Let f, g, h: X → Y be continuous mappings. These functions are said to be

3-Homotopic if ∃ a continuous map F: X × [0, 1] → Y defined such that

(i) F(x, 0) = f(x), (ii) F(x, 1) = g(x) and (iii) F(x, 2) = h(x), ∀x ∈ [0, 1].

Note: If X and Y are general sets, then we should drop the condition of continuity of F.

Example 1: An example is F(x, t) = f(x) + t (2 - t) g(x) - h(x)

Example 2: Let us define a 3-Homotopy with f(x) = x and g(x) = x^2 as a 2-Homotopy. By adding a third curve to the two Homotopy curves which may be obtained by the fact that it is path-Homotopic between (0, 0) and (1, 1) whose general equation is y = a$(x-h)^2$ + k

Thus we have the functions F_t(x) = (1- t) x + t · (x^2) [2] i.e., F_t(x) = (x^2)t – tx + x; where F_t(0, t) = 0 = f(x) and

F_t (1, t) = 1 = g(x) and h(x) are the values in the interval (0, 1) continuously.

2.6. Definition: Group Action

A group (G, f) is said to act on a set X if ∃ a function F: G × X → X such that

(i) F (e, x) = x and (ii) F(f(g_1, g_2), x) = F (g_1, F(g_2 ,x))

We have the convention that g(x) = F(g, x) i.e., 'g' acts on x which we again write as g · x where '·' is the binary operation in G but note that g · x is not a group operation.

IV. RESULT & DISCUSION

In this section we define the concept of Group Action induced by Homotopy.

3.1. Group Action induced by Homotopy:

Let F: G × [0, 1] → [0, 1] be the Homotopy between two functions, f, g: G → [0, 1] where G is a group· Suppose F is also a Group Action then we say that F is an "Induced Group Action".

Thus, if a Homotopy F is also a Group Action then we say that "Group Action is induced by Homotopy".

Theorem 3.1:

Let F: G × [0, 1] → [0, 1] be the Homotopy between two functions, f, g: G → [0, 1] where G is a group. If G is also a Group Action, then it satisfies the conditions:

x · f(y) = f(x, y)

x · g(y) = g(x, y)

Proof: Since F is a Homotopy and also a Group Action we have the following conditions:

$$\left.\begin{array}{l} F(e, x) = x, \ \forall x \in (0, 1) \\ F(h_1 h_2, x) = F(h_1, F(h_2, x)) \end{array}\right\}$$ ---------- since F is a Group Action; and

$$\left.\begin{array}{l} F(h, 0) = f(x) \\ F(h, 1) = g(x) \end{array}\right\}$$ ------------- since F is a Homotopy

We now show that 'f' and 'g' satisfy the following conditions:

(i) x · f(y) = f(x, y)

(ii) x · g(y) = g(x, y)

F($h_1 h_2$, 0) = F(h_1, F(h_2, 0)) -------since F is a Group Action

i.e., f($h_1 h_2$) = F(h_1, f(h_2))

= h_1 · f(h_2) since F is a Group Action i.e., F(g, x) = g(x)

Observe that 'f' satisfies the same condition. Hence in general 'f' must satisfy the condition f(x, y) = x · f(y) &/OR f(x + y) = x + f(y)

Similarly, 'g' also satisfies the condition g(x, y) = x · g(y) as F($h_1 h_2$, 1) = h_1 (g(h_2))

Theorem 3.2(Converse)

Statement: If f, g: G → [0, 1] are two functions such that they satisfy the conditions

x · f(y) = f(x, y)

x · g(y) = g(x, y) and

F: G × [0, 1] → [0, 1] be the Homotopy between the two functions 'f' and 'g', then F is also a Group Action.

Proof: Let F: G × [0, 1] → [0, 1] be the Homotopy between two functions, f, g: G → [0, 1] satisfying the given conditions.

We need to show that: F is a Group Action

Since F is a Homotopy, we have F(x, 0) = f(x) and F(x, 1) = g(x). Also f and g are functions satisfying

(i) x · f(y) = f(x, y)

(ii) x · g(y) = g(x, y), ∀ x ∈ (0, 1)

Consider (e, x)

e · f(x) = f(e, x) = x

x · f(y) = f(x, y)

We define the mapping F: $E^3 \times [0, 1] \rightarrow E^3$ by F(x, y) = x + y. We have F(e, 0) = f(e) ------(a)

$F(g_1 + g_2, x) = g_1 + g_2 + x$ ----------(1)

Also $F(g_1, F(g_2, x)) = F(g_1, g_2 + x) = g_1 + g_2 + x$ -----------(2)

Clearly from (1) and (2), it follows that $F(g_1 + g_2, x) = F(g_1, F(g_2, x))$

Thus, Homotopy along with these additional conditions induces a Group Action. Hence the converse.

Note: On the existence and properties of 'f' and 'g': Any linear function F: G × [0, 1] → [0, 1] such that

F(x + y) = x + f(y) is the identity. It can be shown that non-linear functions exist with the above property by using axiom of choice. Recently without using axiom of choice, one can construct such a function.

Corollary: Properties of the function 'f' such that f(x + y) = x + f(y) are as follows:

This function is controlled by f(0).

Function 'f' has the property of absorbing other elements.

'f' must act on a set to itself under these conditions group actions leads to Homotopy.

V. APPLICATION

The crux of our work is mainly to build a ground for a Unified theory approach establishing the co-existence of Homotopy and Group Action on a single platform. The ease of defining a Homotopic map which in-turn induces a Group Action helps to overcome the challenge of finding Group Actions individually. The need for establishing such a co-existence in the real world will serve as an application in several complex mechanisms present in nature. We try relating the co-existence of Homotopy and Group Actions to problems that can be associated to the complex functioning of the human body especially in the human ear, the human heart or tracking of facial features in the life span of an individual to construct an approximate future appearance after a stipulated time due to ageing (i.e., the Homotopic Group actions facilitate to infer the facial features at an early age/childhood and construct an image of the person's face after passing of time, incorporating the changes over a period of time due to the ageing factor). We aim to find one such real-time application of their co-existence in our future work.

VI. CONCLUSION

This paper mainly focuses on interlinking Group Action and Homotopy. To establish the conditions required we define "Induced Group Action". This paper discusses the properties of a function as conditions for Homotopy leading to group action. The main results, established as theorems, discuss the criteria to be met by a function which acts as a Homotopy as well as a Group Action. We conjecture that additional conditions are essential for a Homotopy to induce a Group Action as defined in this paper. We shall work on extending this further to analyze the possibility of a group Action inducing a Homotopy and the required criteria. The established facts are new ideas which we anticipate to open new horizons to explore in future. Further study may help verify the possibility of interpreting real problems where the presence of either one of Homotopy or Group Action, leads to the other. One may consider finding interpretations of the hearing loss analysis using the idea presented in this paper.

VII. FUTURE SCOPE

In future it is used for advancements in topological data analysis, applications in robotics for motion planning, enhancements in understanding the symmetry properties of mathematical spaces, and potential contributions to quantum computing through the study of topological phases and invariants.

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REFERENCES

[1] J V Deshpande, "Introduction to Topology", Tata McGraw-Hill Publishing Company Limited.
[2] James Munkres, "Topology", Pearson New International Edition, 2015.
[3] Amthul Muqheet, Dr B.R. Srinivasa, "Action of Sets", Journal of Emerging Technologies and Innovative Research (JETIR)-JETIR1904A03 (ISSN-2349-5162), April 2019, Volume 6, Issue 4, page 11-22, http://doi.one/10.1729/Journal.20449.
[4] Richard Earl, "Groups and group Actions", Lecture notes, Hilary and Trinity Terms 2014.
[5] David Blanc and Debasis Sen, "Realizing Homotopy Group Actions", Bull. Belg. Math. Soc. Simon Stevin 21 (2014), 685–710
[6] Span Spanbauer & Luke Sciarappa, "Neural Group Actions", arXiv:2010.03733v1 [stat.ML] 8 Oct 2020.
[7] S.M.S Omer, N.H. Sarmin and A.Erfanian, "The Probability that an Element of a Symmetric Group Fixes a Set and its Application in Graph Theory", World Applied Sciences Journal, DOI: 10.5829/idosi.wasj.2013.27.12.13288
[8] Karen E. Smith, "Group Actions and Orbits", 2018 UM Math Department licensed under a Creative Commons
By-NC-SA 4.0 International License.
[9] Dong Youp Suh, "Group Actions on Homotopy Complex Projective Spaces", Comm. Korean Math. Soc. 5 (1990), No. 1, pp. 129",160
[10] Sandi Klavzar, Tsai-Lien Wong, Xuding Zhu, "Distinguishing labellings of group action on vector spaces and graphs", Journal of Algebra, 2006. DOI: 10.1016/j.jalgebra.2006.01.045
[11] Gary Lee Walls and Marian Deaconescu, "Groups acting on groups", Algebra and Logic, DOI: 10.1007/s10469-013-9250-9
[12] Bartosz Frej and Dawid Huczek, "Extensions of full shifts with Group Actions", arXiv:1901.01145v1 [math.DS] 4 Jan 2019.
[13] Ali Joohy, "Homotopy Analysis Technique for a generalized (1+1)-Dimensional KDV Equation of variable coefficients", arXiv:1904.13306v2 [nlin.PS] 1 May 2019
[14] D.S Oliveira and E. Capelas de Oliveira, " Analytical solutions for Navier-Stokes equations with Ψ-Caputo fractional derivative", arXiv:2005.05917v1 [math.AP] 8 May 2020
[15] Yue Wu, Zuoguan Wang and Qiang Ji, "Facial feature tracking under varying facial expressions and face poses based on restricted Boltzmann machines", arXiv:1709.05731v1 [cs.CV] 18 Sep 2017.
[16] B. Ebenezer, J. K. K. Asamoah, I. Khan, A.-A. S. Kwesi, M. A. Khan, "The multi-step homotopy analysis method for a
modified epidemiological model for measles disease", Nonlinear Sci. Lett. A, 8(2)(2017), 320-332.