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DIFFERENTIATION OF FUNCTIONS

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Abstract : This paper contains differentiation of functions by using the different functions and definition of the derivatives. The generalize results established formulae for various special functions are formulate.

Keywords:- Differentiation, function, derivatives, formula, formulate.

I. Differentiating Sums of Functions

In this paper we focus on functions given by formulas. The derivatives of such functions are then also given by formulas.

$f(x)$ equals $x^{n_1}+x^{n_2}$ which is a sum of two functions of $x, A(x)$ and $B(x)$ where $A(x)=x^{n_1}$ and $B(x)=x^{n_2}$. Therefore, if $f(x)=A(x)+B(x)$. What would $f'(x)$ be ? The answer is that the derivative is the sum of the derivatives of the two functions. To prove this let us return to the definition of the derivative. We can express a small change in f, Δf, equal to $f(x+\Delta x)-f(x)$.

Therefore :

$$\Delta f = f(x+\Delta x)-f(x)=\left[A(x+\Delta x)+B(x+\Delta x)\right]-\left[A(x)+B(x)\right]$$

Δx and taking the limit as Δx goes to zero gives us the instantaneous rate of change of f with respect to x.

$$\frac{df}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{\left[A(x+\Delta x)+B(x+\Delta x)\right]-\left[A(x)+B(x)\right]}{\Delta x}$$

Combing the $A(x)$ and $B(x)$ terms together simplifies the above expression to :

$$\frac{df}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left[A(x+\Delta x)-A(x)\right]}{\Delta x}+\frac{\left[B(x+\Delta x)-B(x)\right]}{\Delta x}$$

Which reduces to :

$$\frac{df}{dx}=f'(x)=A'(x)+B'(x)$$

Therefore if $f(x)$ is a sum of two functions of x, then its derivative with respect to x is the sum of the derivatives of the functions with respect to x. Thus :

$$f(x)=x^{n_1}+x^{n_2}$$

$$\frac{df}{dx}=f'(x)=n_1.x^{n_1-1}+n_2.x^{n_2-1}$$

Similarly if $f(x)$ is defined in terms of a difference among some functions of x, then $f'(x)$ is the sum of the difference among the derivatives of the functions.

$$f(x) = x^{n_1} - x^{n_2} - x^{n_3}......$$

$$\frac{df}{dx} = f'(x) = n_1.x^{n_1-1} - n_2.x^{n_2-1} - n_3.x^{n_3-1}.........$$

II. Differentiating Products of Functions

Consider the following function :

$$f(x) = x^2.(x+1)^3$$

If we let, then $f(x)$ can be expressed as the product of the two function $A(x)$ and $B(x)$ such that :

$$f(x) = A(x).B(x)$$

We can differentiate products of functions by using the definition of the derivative. A small change in f can be written as :

$$\Delta f = f(x+\Delta x) - f(x) = A(x+\Delta x) + B(x+\Delta x) - A(x) + B(x)$$

Next, divide by Δx to calculate the rate of change of f with respect to f :

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{A(x+\Delta x)B(x+\Delta x) - A(x)B(x)}{\Delta x}$$

Taking the limit as Δx goes to zero gives us the instantaneous rate of change of f with respect to x, or the derivative of $f(x)$.

$$\frac{df}{dx} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{A(x+\Delta x)B(x+\Delta x) - A(x)B(x)}{\Delta x}$$

From the definition of the derivative we know that :

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

Multiplying both sides by this infinitely small Δx.

$$f(x+\Delta x) - f(x) = f'(x).\Delta x$$

$$f(x+\Delta x) = f(x) + f'(x).\Delta x$$

Since both $A(x)$ and $B(x)$ are functions of x, then $A(x+\Delta x)$ and $B(x+\Delta x)$ can be substituted with $A(x) + A'(x).\Delta x$ and $B(x) + B'(x).\Delta x$ respectively. Note that this substitution only holds true for Δx going to zero. We now have :

$$f'(x) = \lim_{\Delta x\to 0} \frac{\left[\left(A(x)A'(x).\Delta x\right) + \left(B(x)B'(x).\Delta x\right)\right]}{\Delta x}$$

$$\lim_{\Delta x\to 0} \frac{-\left[A(x)B(x)\right]}{\Delta x}$$

Expanding the numerator :

$$f'(x) = \lim_{\Delta x\to 0} \frac{\left[A(x)B(x) + A(x)B'(x).\Delta x + A'(x)\Delta x B(x)\right]}{\Delta x}$$

$$\lim_{\Delta x\to 0} \frac{+\left[A'(x)B'(x)\Delta x^2\right] - \left[A(x)B(x)\right]}{\Delta x}$$

Canceling terms and dividing through by Δx reduces it to :

$$f'(x) = \lim_{\Delta x\to 0} A(x)B'(x) + B(x)A'(x) + A'(x)B'(x)\Delta x$$

$$\therefore f'(x)A(x)B'(x)+B(x)A'(x)$$

Thus the derivative of a function $f(x)$ that is a product of two functions of x, is simply the product of the first function and the derivative of the second function plus the product of the second function and the derivative of the first function.

$$f(x)=A(x).B(x)$$

$$f'(x)=A(x).B'(x)+B(x)A'(x)$$

III. Differentiating Functions of any Power N

We have proven that if $f(x)=x^n$ then $f'(x)=n.x^{n-1}$ for n equal to a positive integer, i.e. 1, 2, 3, 4 etc. What if n were a faction such that $f(x)=x^{\frac{1}{3}}$. As we shall prove, the derivative of any function of *x* of the form $f(x)=x^n$ is always $f'(x)=n.x^{n-1}$ where *n* is any real number, positive, negative, or fraction. This is no coincidence but is because of the way exponents are defined as a continuous operation for any *n*.

Let us consider the first case of $f(x)=x^{\frac{1}{n}}$, where *n* is any positive integer and $1/n$ is a fraction.

$$f(x)=x^{\frac{1}{n}}$$

$$f=x^{\frac{1}{n}}$$

Raising both sides to the *n'th* power :

$$f^n=x$$

$$x=f^n$$

Differentiating *x*, with respect to *f* yields :

$$f^n=x$$

$$x=f^n$$

Taking the reciprocal of the function :

$$\frac{dx}{df}=\frac{1}{n.f^{n-1}}=\frac{1}{n}.\frac{1}{f^{(n-1)}}$$

From the definition of the function, we know that $f=x^{\frac{1}{n}}$. Making this substitution :

$$\frac{df}{dx}=\frac{1}{n}.\frac{1}{x^{\frac{1}{n}(n-1)}}$$

$$\frac{df}{dx}=\frac{1}{n}.\frac{1}{x^{\left(1-\frac{1}{n}\right)}}$$

$$\frac{df}{dx}=\frac{1}{n}.x^{-\left[\left(1-\frac{1}{n}\right)\right]}$$

$$\therefore \frac{df}{dx}=f'(x)=\frac{1}{n}.x^{\frac{1}{n}-1}$$

This concludes the proof that if $f(x)=x^n$ then $f'(x)=n.x^{n-1}$, for any positive *n*, integer or fraction. We will prove it also holds true for *n* as a negative number.

We can use the product rule to prove that the derivative of $f(x)=x^n$ is $f'(x)=n.x^{n-1}$ for all *n*, negative real numbers also. Consider the function :

$$f(x)=x^{-n}$$

$$f=\frac{1}{x^n}$$

Multiplying both sides by x^n yields :

$$f.x^n=1$$

$$f(x).x^n = 1$$
$$f(x).Ax = B(x)$$

where $Ax = x^n$ and $B(x) = 1$

The left side of the equation represents a product of two functions, f and x^n, and the right side is a constant function, $B(x) = c = 1$.

Since both functions are equal to each then their derivatives must be equal. Using the product rule to differentiate the right side with respect to x results in :

$$\frac{d\left[f(x).A(x)\right]}{dx} = f(x).A'(x) + A(x).f'(x)$$

Similarly, differentiating the right side with respect to x yields :

$$B(x) = 1.x^0$$
$$B'(x) = 0$$

Setting the derivatives of the left and right side equal to each other :

$$f(x).A'(x) + A(x).f'(x) = 0$$

Remember that :

$$f(x) = x^{-n} = \frac{1}{x^n}$$
$$A(x) = x^n$$
$$A'(x) = n.x^{n-1}$$

Making the required substitutions :

$$\left(\frac{1}{x^n}\right).nx^{n-1} + x^n.f'(x) = 0$$
$$\frac{n}{x} + x^n.f'(x) = 0$$

We can now solve for $f'(x)$

$$f'(x) = -\left(\frac{n}{x}\right).\frac{1}{x^n}$$
$$f'(x) = -n.\frac{1}{x^{1+n}}$$
$$f'(x) = -n.x^{-(1+n)}$$
$$f'(x) = -n.x^{-n+1}$$

Therefore if $f(x) = x^{-n}$ then $f'(x) = -n.x^{-n+1}$, where $-n$ is a negative integer.

We have proven that if $f(x) = x^n$ is $f'(x) = -n.x^{n+1}$ for positive real numbers and negative integers. What remains is to prove it true for negative fractional powers as well. To do this let,

$$f(x) = x^{\frac{1}{n}}$$
$$f(x) = \frac{1}{x^{\frac{1}{n}}}$$
$$f(x) = x^{\frac{1}{n}}$$

Differentiating both sides with respect to x yields :

$$f(x).\frac{1}{n}.x^{\frac{1}{n}-1} + x^{\frac{1}{n}}.f'(x) = 0$$

Substituting known values for $f(x)$ and solving $f'(x)$ for gives us :

$$f'(x) = -\frac{1}{n}.x^{\frac{1}{n}-1}$$

Thus if $f(x)=x^{-n}$ then $f'(x)=-n.x^{-n+1}$ for any n, positive or negative real number.
This is true because of the way exponents are defined as a uniform operation for any n.

The purpose of going through all the proofs for the different cases of n was to give you a better understanding of how to differentiate functions of x with respect to x,

IV. Differentiating Functions of Functions

The last technique of differentiation is for differentiating functions of functions of x or functions of the form, $f(g(x))=g(x)^n$.

For example consider the function : $f(x)=(x^2+1)^{\frac{1}{2}}$

The derivative is not $f'(x)=\frac{1}{2}(x^2+1)^{\frac{1}{2}-1}$ because we also need to take into consideration the inside function of x, x^2+1. We can replace x^2+1 with $g(x)$ and get :

$$f(g(x))=g(x)^{\frac{1}{2}}$$

$$f(g)=g^{\frac{1}{2}}$$

To find the derivative of $f(g(x))$ with respect to x, we first need to find the derivative of f with respect to g. From the definition of the derivative :

$$\frac{df}{dg}=f'(g)=\lim_{\Delta g\to 0}\frac{f(g+\Delta g)-f(g)}{\Delta g}$$

Next we find the derivative of $g(x)$, the inside function, with respect to x.

$$\frac{dg}{dx}=g'(x)=\lim_{\Delta r\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

Now we multiply the two derivatives $f'(g)$ and $g'(x)$ to get df/dx :

$$f'(g).g'(x)=\frac{df}{dg}.\frac{dg}{dx}$$

$$=\lim_{\Delta r\to 0}\frac{f(g+\Delta g)-f(g)}{\Delta x}.\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$\Delta g=g(x+\Delta x)-g(x)\, f'(g)g'(x)$$

$$=\frac{df}{dg}.\frac{dg}{dx}=\lim_{\Delta x\to 0}\frac{f(g+\Delta g)-f(g)}{g(x+\Delta x)-g(x)}.\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

Δx goes to zero, $\frac{g(x+\Delta x)-g(x)}{\Delta x}g'(x)$ or : $\frac{dg}{dx}=g'(x)$

Similarly, Δg also goes to zero as Δx goes to zero. Multiplying both sides by dx.

$$dg=g'(x).dx$$

Thus as dx goes to zero;

$$dg=c.0=0$$

As dx goes to zero, $\frac{f(g+\Delta g)-f(g)}{\Delta g}$ becomes $f'(g)$. At the same time the in the denominator of

$$\frac{f(g+\Delta g)-f(g)}{\Delta g}$$

cancels out with Δg the in the numerator of,

$$\frac{g(x+\Delta x)-g(x)}{\Delta x},$$

since they are both the equivalent. To conclude

$$\frac{df}{dg} \cdot \frac{dg}{dx} = \frac{df}{dx} = f'(g).g'(x)$$

Thus the derivative of a function of a function of x with respect to x, is the derivative of the outer function with respect to the inner function, multiplied by the derivative of the inner function with respect to x. Some examples will show how this is done.

V. Using the First Derivative

From the definition of the derivative, $f'(x)$ is the derivative of a function $f(x)$. The derivative of a function tells us how fast f is changing relative to the independent variable, x.

Thus the derivative refers specifically the rate of change of the anti-derivative function with respect to x . Rate of change is also synonymous with the slope of the tangent line to the graph at a particular point. Therefore, a function and its derivative are closely related and knowing just the derivative can tell us a great deal about the behaviour of its anti-derivative.

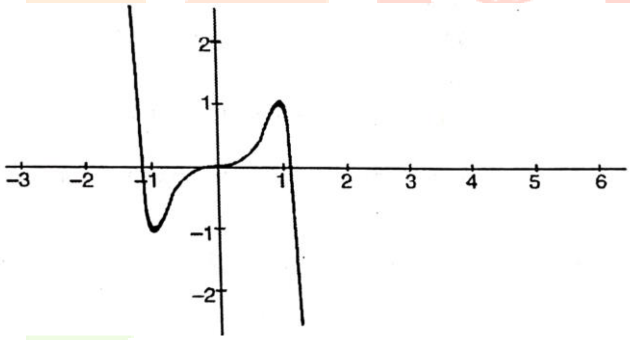
Since the derivative of a function is derived from the definition of a derivative as :

$$\lim_{\Delta r \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We can work with the definition to find the anti-derivative when only the derivative is known. Also recall how differentiation is based on a limiting or subtracting process and dividing; therefore working backwards would tell us that we should be adding and multiplying. We will develop on this later, however, let us first look at how we can use $f'(x)$ to obtain equilibrium points on the graph of the anti-derivative or $f(x)$.

Equilibrium points are by definition, points on the graph refer to static situations where the rate of change is zero. Thus a change in the independent variable results in no change in the dependent variable.

Equilibrium rate generally occurs when a situation has reached a critical maximum value and then decreases or where a situation has reached a critical minimum value then increases.

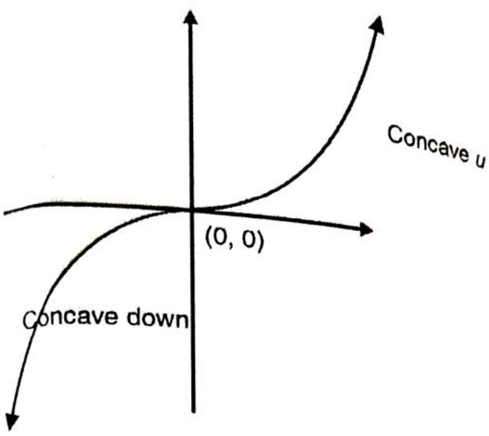


Equilibrium and critical values of a function can refer to different things depending on the phenomena being studied. Therefore we will restrict outselves to the geometric interpretation of an equilibrium as a point on the graph where the rate of change is zero.

Since the rate of change is zero, the tangent to the graph at this point will be a hori-zontal line.

A horizontal tangent tells us that the derivative's value at the equilibrium point is zero.

Such situations occur when the graph has reached a maximum or minimum value. Horizontal tangent may also exist, but not necessarily, when the concavity of a graph changes.



A change in concavity occurs at points on the graph called inflection points. As we shall soon study, the derivatives value at an inflection does not have to be zero.

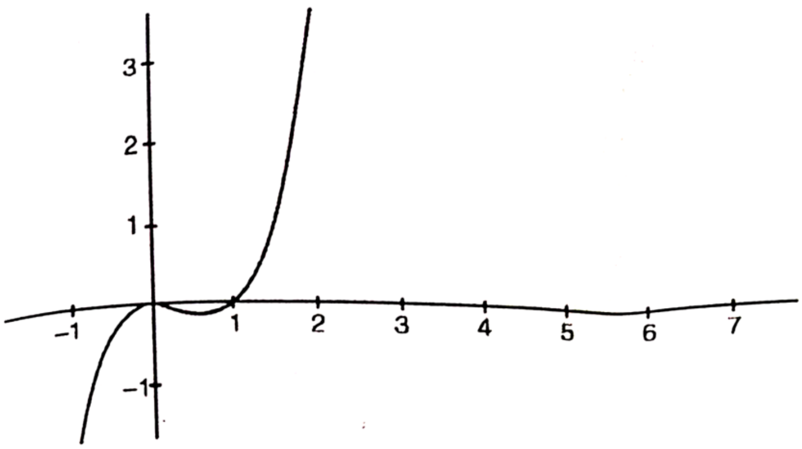
Therefore we will restrict our definition of equilibrium point to reflect either maximum or minimum values on the graph.

For example to find equilibrium points for the function $f'(x)=x^3-x^2$, we first need to differentiate it to get $f'(x)$. The derivative $f'(x)$, tells us the instantaneous rate of change of function, $\frac{df}{dx}$ at any point x.

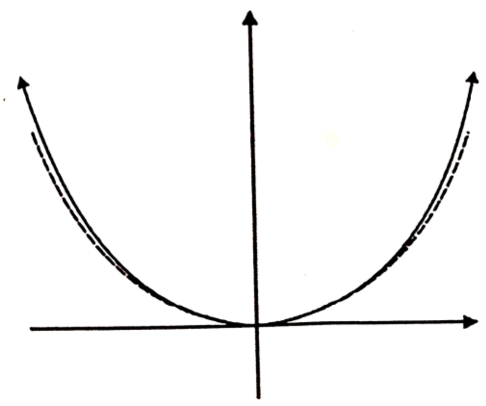
Since we want to find points where the rate of change of $f(x)$ is zero, we need to set equal to zero to find those values of x which the satisfy the equation, $f'(x)=0$. Doing this for $f(x)=x^3-x^2$ results in :

$$f'(x)=3x^2-2x$$
$$f'(x)=0$$
$$3x^2-2x=0$$
$$(x)(3x-2)=0$$
$$\therefore x=0 \text{ and } x=2/3$$

This tells us that at both $x=0$ and $x=2/3$ there exists as equilibrium point, which is confirmed by the graph of the function $f(x)=x^3-x^2$.



Having found the critical points, how do we classify them as either maximums or minimums? Obviously this can be done by just looking at the graph of $f(x)$, but the purpose of this chapter is to understand how only use the derivative, $f'(x)$ to approximate the behaviour of the function $f(x)$. To determine whether our equilibrium points are either maximum or minimums we need to evaluate points left and right of the equilibrium points to determine where the function $f(x)$, is increasing or decreasing.

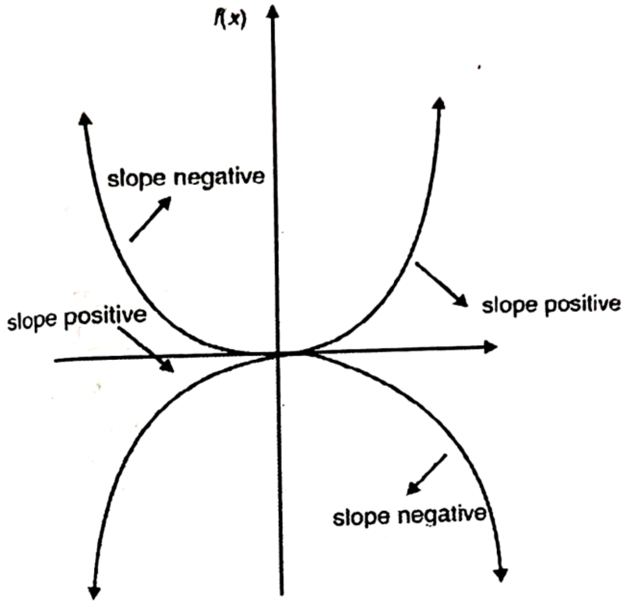


A minimum is defined as the bottom of a U-shaped or concave up graph. If the graph is concave up then the slope or rate of change is positive to the right of an equilibrium point and the function is increasing to the right of that point.

To the left of the equilibrium point, the slope is negative which means the function is decreasing till it reaches the equilibrium point. To better understand this, look at the following graph of a concave up portion of a graph. Notice how the function decreases till it reaches the equilibrium point then rises after passing it.

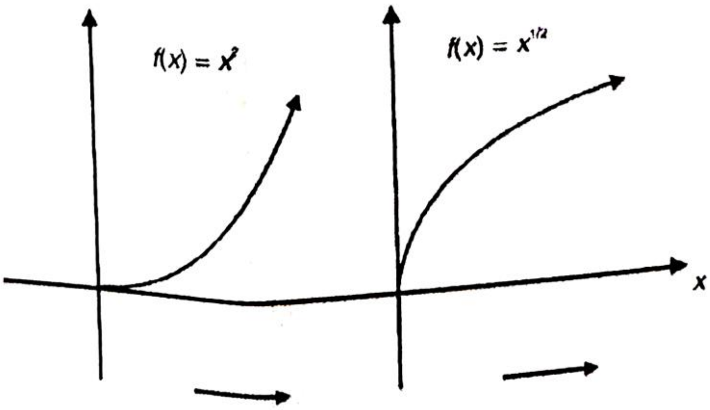
Similarly in the rate of change of $f(x)$ where negative on the right side and positive on the left side of the equilibrium point, then we get an inverted U shaped or concave down graph.

A concave down graph thus reflects a maximum value at the equilibrium point. If the slope is both positive or negative on either side of the equilibrium point then we get an inflection point that represents where the concavity changes from a U shape to a $\cap$ shape or vice-versa. The following graph summarizes the above conclusions.



VI. Using the Second Derivative

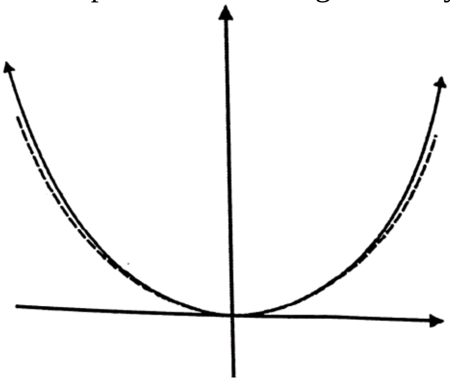
The first derivative allows us to define equilibrium points on the graph of a function $f(x)$. By evaluating points to the left and right of the equilibrium point we can classify these points as either maximums or minimums and thus determine the concavity of the graph. Without having found the equilibrium points it is extremely difficult to determine the behavior of a function over an interval.



The sign of the first derivative only tells us if a function is increasing or decreasing; however, a function can increase or decrease in two way. For example consider the graphs of the following two different functions. In both cases the function is increasing and the first derivative is always positive; however each function increases in a different way i.e. one increases concave up and the other increases concave down. Using the first derivative only, we would have to know not only where its positive or negative but also how the first derivative is changing i.e. positive and increasing, negative and increasing etc.

For the first graph $f''(x)$ is positive and increasing thus the graph of $f(x)$ is increasing and concave. For the second graph, $f''(x)$ is also positive but is decreasing. Thus the graph of $f(x)$ is concave down. The process of looking only at the graph of the first derivative to understand how $f(x)$ behaves is an extremely abstract and difficult one.

To quicken and simplify our work we can use the function's second derivative to conclude where the graph is concave up or down. This information along with the fact that the derivative is either positive or negative over an interval will be enough to accurately determine a function's behavior. Recall that a property of a concave up part of a graph is that its slope or rate of change is always increasing.



On the left side the slope is negative; however, as x increases the slope gets less and less, -5, -3, -2 till it reaches 0, from where on it increases to 1, 5, 6 etc. We can then conclude that the rate at which the slope is changing must be positive or the graph of the derivative is increasing. Since the derivative's value is constantly increasing, then the rate of change of the derivative, given by $f''(x)$ will be positive. Remember that positive rate of change implies that the function is increasing over that interval, while a negative rate of change implies the function decreases as x increases.

In a concave up graph the derivative is increasing, such that the second derivative over this interval will be positive. Working in reverse we arrive at an important conclusion. If the second derivative is positive over ran interval, then the first derivative is increasing, implying that the graph of the original function, $f(x)$ is concave up. This is true because the rate of change of a concave up graph is always increasing.

The reverse is true for concave down graphs. If the second derivative is negative then the first derivative is decreasing, implying that the original functions graph is concave down over the interval, $f''(x)>0$ is concave up $f''(x)<0$ hence $f(x)$ is concave down. Though all the information concerning the behavior of $f(x)$ can be obtained from studying its derivative, we can quicken and confirm our sketches by looking at the functions second derivative.

Without having found any equilibrium points we can accurately determine the behavior of $f(x)$ over an interval by using the signs of both the function's first and second derivative simultaneously. Four possibilities may exist for the signs of the derivatives. Both $f'(x)$ and $f''(x)$ are positive over an interval.

1. Therefore $f(x)$ is increasing and concave up.
2. $f'(x)$ is positive but $f''(x)$ is negative. Thus $f(x)$ is increasing and concave down
3. $f'(x)$ and $f''(x)$ are negative, in which case $f(x)$ is decreasing and concave down.
4. $f'(x)$ is negative but $f''(x)$ is positive, thus $f(x)$ is decreasing and concave up.

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