IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Bayesian Approximation Of The Location Parameter Of Generalized Compound Rayleigh Distribution With Linex Loss

Prof. Uma Srivastava & Parul Yadav

Department of Mathematics & Statistics,

DDU Gorakhpur University,

Gorakhpur - 273009 (India).

Abstract

The paper provides the Bayes estimate of shape parameter of Generalized Compound Rayleigh distribution assuming rest two scale and location parameter as known under the Linex loss function. Then by using Lindley approximation procedure we have obtained the Approximate Bayes estimate of Location parameter of Generalized Compound Rayleigh distribution under the Linex loss functions. We have done the sensitivity analysis of the Approximate Bayes estimators of model when prior specifications deviate from the true values and presented a numerical study to illustrate the above technique on generated observations and numerical comparison is done by R-programming.

Keywords: MLE and Bayes Estimator, Lindley Approximation,Generalized Compound Rayleigh distribution, Linex loss function, Approximate Bayes estimate.

1. INTRODUCTION

The Generalized Compound Rayleigh Distribution is a special case of the three-parameter Burr type XII distribution. Mostert, Roux, and Bekker (1999) considered a gamma mixture of Rayleigh distribution and obtained the compound Rayleigh model with unimodal hazard function. This unimodal hazard function is generalized and a flexible parametric model is thus constructed, which embeds the compound Rayleigh model, by adding shape parameter. $f(x;\alpha,\beta,\gamma) = \alpha\gamma\beta^{\gamma}x^{\alpha-1}(\beta+x^{\alpha})^{-(\gamma+1)} \quad x,\alpha,\beta,\gamma > 0$ (1.1)

With Probability Distribution Function

$$\mathrm{F(x)} = 1-(1-\beta x^{\alpha})^{-\gamma} \qquad \mathrm{x},\alpha,\beta,\gamma > 0 \qquad (1.2)$$

Reliability function **is**

$$\mathrm{R(t)} = \left(\frac{\beta}{\beta+t^{\alpha}}\right)^{\gamma}$$

Hazard rate function

$$\mathrm{H(t)} = \alpha\gamma\frac{t^{\alpha-1}}{\beta+t^{\alpha}}$$

The Generalized compound Rayleigh model includes various well-known pdfs, namely

(i) Beta-Prime pdf , if $\alpha=\beta=1$

(ii) $\alpha = 1$

(iii) Burr XII pdf(Burr,1942), if $\beta = 1$

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta},\theta) = k(\hat{\theta}-\theta)^2$ where $\hat{\theta}$ is the estimate of θ, the loss function is called quadratic weighed loss function if k=1, we have

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \qquad (1.3)$$

Known as squared error loss function (SELF). This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Canfield (1970), Basu and Ebrahimi(1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function known as Linex Loss Function for a number of distributions. Such as a loss function is derived as

$$L(\Delta) = b\exp(a\Delta) - c\Delta - b \qquad (1.4)$$

Where $\Delta = (\hat{\theta} - \theta)$

And $a, c \neq 0,\ b > 0$

The underestimate of the failure rate results in more serious consequences than an overestimation of the failure rate. This leads to the statistician to think about asymmetrical loss function which has been proposed in statistical literature. Ferguson (1967), Zellner & Geisel (1968), Rojo (1987), Aitchision & Dunsmore (1975) and Berger (1980) have considered the linear asymmetric loss function. Varian (1975) introduced the following convex loss function known as LINEX. (Linear Exponential) Loss Function i.e. given as;

$L(\Delta) = be^{a\Delta} - c\Delta - b\,;\ a, c \neq 0, b > 0$

Where $\Delta = \hat{\theta} - \theta$. It is clear that L(0) = 0 and the minimum occurs when ab=c, therefore , $L(\Delta)$ can be written as

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1],\ a \neq 0, b > 0; \qquad (1.5)$$

Where a and b are the parameters of the loss function may be defined as shape and scale respectively. The loss function has been considered by Zellner (1986) , Rojo (1987) , Basu and Ebrahimi (1991) considered the $L(\Delta)$ as

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1],\ a \neq 0, b > 0 \qquad (1.6)$$

Where, $\Delta = \frac{\hat{\theta}}{\theta} - 1$

2. The Estimators

Let $x_1 \leq x_2 \leq \ldots\ldots\ldots \leq x_n$ be the n failures in complete sample case. The likelihood function is given by;

$$L(\underline{x} \mid \alpha, \beta, \gamma) = \prod_{j=1}^{n} f(x_j, \alpha, \beta, \gamma)$$

$$= \alpha^n \gamma^n \beta^{n\gamma} \prod_{j=1}^{n} x_j^{\alpha-1} \prod_{j=1}^{n} (\beta + x_j^{\alpha})^{-(\gamma+1)}$$

$$L(\underline{x} \mid \alpha, \beta, \gamma) = (\alpha\gamma)^n U e^{-\gamma T} \qquad (2.1)$$

where

$T = \sum_{j=1}^{n} log\left[1 + \frac{x_j^{-\alpha}}{\beta}\right]$ and $U = \prod_{j=1}^{n} \frac{x_j^{\alpha-1}}{(\beta + x_j^{\alpha})}$

from equation(2.1) the log likelihood function is

$$Log\ L = n\ log\alpha + nlog\gamma + n\gamma\ log\beta + (\alpha - 1)\sum_{j=1}^{n} logx_j$$

$$-(\gamma + 1)\sum_{j=1}^{n} (\beta + x_j^{\alpha}) \qquad (2.2)$$

and differentiation of equation(2.2) with respect to α, β and γ yields respectively we get

$$\frac{\partial\, Log\, L}{d\alpha} = \frac{n}{\alpha} + \sum_{j=1}^{n} log x_j - \sum_{j=1}^{n} \frac{x_j{}^{\alpha} log x_j}{\beta + x_j{}^{\alpha}} - \gamma \sum_{j=1}^{n} \frac{x_j{}^{\alpha} log x_j}{\beta + x_j{}^{\alpha}} \qquad (2.3)$$

$$\frac{\partial\, Log\, L}{\partial\beta} = \frac{n\gamma}{\beta} - (\gamma+1) \sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}} \qquad (2.4)$$

can also be written as

$$\frac{\partial\, Log\, L}{\partial\beta} = -\sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}} + \gamma \sum_{j=1}^{n} \frac{x_j{}^{\alpha}}{\beta(\beta + x_j{}^{\alpha})} \qquad (2.5)$$

$$\frac{\partial\, Log\, L}{\partial\gamma} = \frac{n}{\gamma} + n \log\beta - \sum_{j=1}^{n} log\left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) - n\, log\, \beta$$

$$= \frac{n}{\gamma} - \sum_{j=1}^{n} log\left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) \qquad (2.6)$$

setting the expressions for the derivatives in 8 equal to zero and solving α, β and γ yield. The maximum likelihood estimators (MLE) of the parameters namely $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{\gamma}_{MLE}$.

However, no closed form solutions exist in this case the elimination of γ in $\frac{\partial\, Log\, L}{\partial\beta}$ and $\frac{\partial\, Log\, L}{\partial\alpha}$ and in $\frac{\partial\, Log\, L}{\partial\alpha}$ and $\frac{\partial\, Log\, L}{\partial\gamma}$ yield a set of equations in terms of β and α.

$$\frac{\sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}}}{\sum_{j=1}^{n} \frac{x_j{}^{\alpha}}{\beta + x_j{}^{\alpha}}} - \frac{n}{\sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \qquad (2.7)$$

and

$$\frac{n}{\alpha} + \sum_{j=1}^{n} \log x_j - \Sigma \frac{x_j{}^{\alpha} log x_i}{(\beta + x_j{}^{\alpha})} - \frac{n \sum_{j=1}^{n} \frac{x_j{}^{\alpha} log x_i}{(\beta + x_j{}^{\alpha})}}{\sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \qquad (2.8)$$

respectively. Applying the Newton-Raphson method $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ can be derived and then from them $\hat{\gamma}_{MLE}$ can be obtained.

3. Bayes estimate for γ with known parameter α, β.

If $\hat{\alpha}$ and $\hat{\beta}$ is known we assume $\gamma(a, b)$ as conjugate prior for γ as

$$g(\gamma|\underline{x}) = \frac{b^a}{\Gamma a} \gamma^{a-1} e^{-\gamma b}\,; \quad (a, b) > 0, \gamma > 0 \qquad (3.1)$$

combining the likelihood function equation(2.1)and prior density equation(3.1) we obtain the posterior density of γ in the form

$$h(\gamma|\underline{x}) = \frac{\alpha^n \gamma^n \beta^{\gamma n} \prod_{j=1}^{n} x_j{}^{\alpha-1} \prod_{j=1}^{n} (\beta + x_j{}^{\alpha})^{-(\gamma+1)} \frac{b^a}{\Gamma a} \gamma^{a-1} e^{-b\gamma}}{\int_0^{\infty} \alpha^n \gamma^n \beta^{\gamma n} \prod_{j=1}^{n} x_j{}^{\alpha-1} \prod_{j=1}^{n} (\beta + x_j{}^{\alpha})^{-(\gamma+1)} \frac{b^a}{\Gamma a} \gamma^{a-1} e^{-b\gamma}\, d\gamma} \qquad (3.2)$$

$$h(\gamma|\underline{x}) = \frac{\gamma^{n+a-1} e^{-\gamma(b+T)}}{\int_0^{\infty} \gamma^{n+a-1} e^{-\gamma(b+T)} d\gamma}$$

Assuming;

$$\sum_{j=1}^{n} \log\left(1+\frac{{x_j}^{\alpha}}{\beta}\right) = T$$

$$h(\gamma|\underline{x}) = \frac{\gamma^{n+a-1}e^{-\gamma(b+T)}(b+T)^{n+a}}{\Gamma(n+a)} \quad (3.3)$$

Bayes Estimator under Linex Loss Function

Let the loss function L(Δ) is

$$L(\Delta) = e^{-k\Delta} - k\Delta - 1; \qquad \mathrm{k} = 0 \quad (3.4)$$

Where

$$\Delta = (\hat{u} - u) \qquad where\ \ u = u(\alpha,\beta,\gamma)$$

Now

$$\mathrm{E}\big(\mathrm{L}(\Delta)\big) = \mathrm{E}\big(e^{k(\hat{u}-u)} - \mathrm{k}(\hat{u}-u) - 1\big)$$

$$\Rightarrow \hat{u}_{ABL} = -\frac{1}{k}\log E_{\mu}\left(e^{-k\hat{u}}\right) \quad (3.5)$$

The Bayes estimator under Linex loss is given by

$$\Rightarrow \hat{\gamma}_{ABL} = -\frac{1}{k}\log\big(E_h(e^{-k\gamma})\big)$$

Now

$$E_h(e^{-k\gamma}) = \frac{(b+T)^{(n+a)}}{\Gamma(n+a)}\int_0^{\infty} e^{-\gamma(k+b+T)}\gamma^{(n+a)}d\,\gamma \quad (3.6)$$

Assuming $\gamma(k+b+T) = y$

$$\hat{\gamma}_{BL} = \frac{(n+a)}{k}log\left(1+\frac{k}{(b+T)}\right) \quad (3.7)$$

4. Approximate Bayes Estimators with unknown $\boldsymbol{\alpha},\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$

Joint prior density α,β,γ is given by

$$\mathrm{G}\ (\alpha,\beta,\gamma) = g_1(\alpha)g_2(\beta)g_3(\gamma|\beta)$$

$$= \frac{c}{\delta\Gamma\xi}\beta^{-\xi}\gamma^{\xi-1}exp\left[-\left(\frac{\gamma}{\beta}+\frac{\beta}{\delta}\right)\right] \quad (4.1)$$

where

$$g_1(\alpha) = c \quad (4.2)$$

$$g_2(\beta) = \frac{1}{\delta}e^{-\frac{\beta}{\delta}} \quad (4.3)$$

$$g_3(\gamma) = \frac{1}{\Gamma\xi}\beta^{-\xi}\gamma^{\xi+1}e^{-\frac{1\gamma}{\beta}} \quad (4.4)$$

The Joint posterior with likelihood equation (3.3) and joint prior equation (4.1)

$$h^*(\alpha,\beta,\gamma) = \frac{\beta^{-\xi}\gamma^{\xi-1}exp\left[-\left(\frac{\gamma}{\beta}+\frac{\beta}{\delta}\right)\right].L(\underline{x}|\alpha,\beta,\gamma)}{\int_\alpha \int_\beta \int_\gamma \beta^{-\xi}\gamma^{\xi-1}exp\left[-\left(\frac{\gamma}{\beta}+\frac{\beta}{\delta}\right)\right].L(\underline{x}|\alpha,\beta,\gamma)d\alpha d\beta d\gamma} \qquad (4.5)$$

The approximate Bayes estimators are evaluated as

$\cup(\Theta) = \cup(\alpha,\beta,\gamma)$

$$\hat{\cup}_{BS} = E(\cup|\underline{x}) = \frac{\int_\alpha \int_\beta \int_\gamma \cup(\alpha,\beta,\gamma)G^*(\alpha,\beta,\gamma)d\alpha d\beta d\gamma}{\int_\alpha \int_\beta \int_\gamma G^*(\alpha,\beta,\gamma)d\alpha d\beta d\gamma} \qquad (4.6)$$

Lindley Approximation Procedure

The ratio of integrals in equation (4.6) does not seem to take a closed form so we must consider the Lindley approximation procedure as

$$E(\mu(\theta,p)|\underline{x}) = \frac{\int \mu(\theta).e^{(l(\theta)+\rho(\theta))}d\theta}{\int e^{(l(\theta)+\rho(\theta))}.d\theta} \qquad (4.6a)$$

Lindley developed approximate procedure for evaluation of posterior expectation of $\mu(\theta)$. Several other authors have used this technique to obtain Bayes estimators (see Sinha(1986), Sinha and sloan(1988),Soliman(2001)).The posterior expectation of Lindley approximation procedure to evaluate of $\mu(\theta)$ in equation (4.6a and 4.6) under SELF, where where $\rho(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function (Lindley (1980)).
The modified form of equation (4.6) is given by

$$E(\cup(\alpha,\beta,\gamma|\underline{x}) = \cup(\Theta) + \frac{1}{2}(A\,\cup_1\sigma_{11} + \cup_2\sigma_{12} + \cup_3\sigma_{13}) + B(\cup_1\sigma_{21} + \cup_2\sigma_{22} + \cup_3\sigma_{23}) + P(\cup_1\sigma_{31} + \cup_2\sigma_{32} + \cup_3\sigma_{33})] + (\cup_1 a_1 + \cup_2 a_2 + \cup_3 a_3 + a_4 + a_5) + 0\left(\frac{1}{n^2}\right) \qquad (4.7)$$

Evaluated at MLE = $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ where

$$a_1 = \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13} \qquad (4.8)$$

$$a_2 = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23} \qquad (4.9)$$

$$a_3 = \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33} \qquad (4.10)$$

$$a_4 = \cup_{12}\sigma_{12} + \cup_{13}\sigma_{13} + \cup_{23}\sigma_{23} \qquad (4.11)$$

$$a_5 = \frac{1}{2}(\cup_{11}\sigma_{11} + \cup_{22}\sigma_{22} + \cup_{33}\sigma_{33}) \qquad (4.12)$$

and

$$A = [\sigma_{11}l_{111} + \sigma_{12}l_{121} + 2\sigma_{13}l_{131} + 2\sigma_{23}l_{231} + \sigma_{22}l_{221} + \sigma_{33}l_{331}] \qquad (4.13)$$

$$B = [\sigma_{11}l_{112} + 2\sigma_{12}l_{122} + 2\sigma_{13}l_{132} + 2\sigma_{23}l_{232} + \sigma_{22}l_{222} + \sigma_{33}l_{332}] \qquad (4.14)$$

$$P = [\sigma_{11}l_{113} + 2\sigma_{13}l_{133} + 2\sigma_{12}l_{123} + 2\sigma_{23}l_{233} + \sigma_{22}l_{223} + \sigma_{33}l_{333}] \qquad (4.15)$$

To apply Lindley approximation on equation (4.6) we first obtain

$$\sigma_{ij} = [-l_{ijk}]^{-1} i,j,k = 1,2,3$$

Likelihood function from Likelihood function(2.2)

$$L = \alpha^n \gamma^n \beta^{n\gamma} \prod_{j=1}^{n} {x_j}^{\alpha-1} \prod_{j=1}^{n} \left(\beta + {x_j}^{\alpha}\right)^{-(\gamma+1)}; (x, \beta, \gamma > 0)$$

$$Log\ L = n\ log\alpha + nlog\gamma + n\gamma log\beta + (\alpha - 1)\sum_{j=1}^{n} \log x_j - (\gamma + 1)\sum_{j=1}^{n} \log(\beta + {x_j}^{\alpha})$$

Now differentiating log likelihood function with respect to α

$$l_1 = \frac{n}{\alpha} + \sum_{j=1}^{n} \log x_j - (\gamma + 1)\omega_{11} \quad \text{where} \quad \omega_{11} = \sum_{j=1}^{n} \frac{{x_j}^{\alpha} log x_j}{\beta + {x_j}^{\alpha}} \tag{4.16}$$

Again differentiating log likelihood function with respect to β

$$l_2 = \frac{n\gamma}{\beta} - (\gamma + 1)\delta_{11} \quad \text{where} \quad \delta_{11} = \sum_{j=1}^{n} \frac{1}{\beta + {x_j}^{\alpha}} \tag{4.17}$$

Again differentiating log likelihood function with respect to γ

$$l_3 = \frac{n}{\gamma} + n \log\beta - q_{11} \quad \text{where} \quad q_{11} = \sum_{j=1}^{n} \log\left(\beta + {x_j}^{\alpha}\right) \tag{4.18}$$

Again differentiating l_1 with respect to α

$$l_{11} = \frac{-n}{\alpha^2} - \beta(\gamma + 1)\omega_{122} \quad \text{where} \quad \omega_{122} = \sum_{j=1}^{n} \frac{{x_j}^{\alpha}(\log x_j)^2}{(\beta + {x_j}^{\alpha})^2} \tag{4.19}$$

Again differentiating l_2 with respect to β

$$l_{22} = \frac{-n\gamma}{\beta^2} - (\gamma + 1)\delta_{12} \quad \text{where} \quad \delta_{12} = \sum_{j=1}^{n} \frac{1}{(\beta + {x_j}^{\alpha})^2} \tag{4.20}$$

Again differentiating l_3 with respect to γ

$$l_{33} = \frac{\partial^2 L}{\partial \gamma^2} = \frac{-n}{\gamma^2} \tag{4.21}$$

Now differentiating l_1 with respect to β

$$l_{12} = (\gamma + 1)\omega_{14} \quad \text{where} \quad \omega_{14} = \Sigma \frac{{x_j}^{\alpha} \log x_j}{(\beta + x_j)^2} \tag{4.22}$$

Again differentiating l_2 with respect to α

$$l_{21} = (\gamma + 1)\omega_{14} \tag{4.23}$$

Again differentiating l_1 with respect to γ

$$l_{13} = -\sum_{j=1}^{n} \frac{{x_j}^{\alpha} \log x_j}{\beta + {x_j}^{\alpha}} = -\omega_{11} \tag{4.24}$$

Again differentiating l_3 with respect to α

$$l_{31} = \frac{\partial^2 L}{\partial \gamma \partial \alpha} = -\sum_{j=1}^{n} \frac{{x_j}^{\alpha} \log x_j}{\beta + {x_j}^{\alpha}} = -\omega_{11} \tag{4.24a}$$

Again differentiating l_2 with respect to γ

$$l_{23} = \frac{n}{\beta} - \sum_{j=1}^{n} \frac{1}{\beta + {x_j}^{\alpha}} = \frac{n}{\beta} - \delta_{11} \tag{4.25}$$

Again differentiating l_3 with respect to β

$$l_{32} = \frac{n}{\beta} - \sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}} = \frac{n}{\beta} - \delta_{11} \tag{4.26}$$

Again differentiating l_{11} with respect to α

$$l_{111} = \frac{2n}{\alpha^3} + (\gamma + 1)\beta\ \omega_{133}\ \text{,where}\quad \omega_{133} = \sum_{j=1}^{n} \frac{x_j{}^{\alpha}(\log x_j)^3(\beta - x_j{}^{\alpha})}{(\beta + x_j{}^{\alpha})^3} \tag{4.27}$$

Again differentiating l_{22} with respect to β

$$l_{222} = \frac{2n\gamma}{\beta^3} - 2(\gamma + 1)\ \delta_{13} \qquad \text{where} \qquad \delta_{13} = \sum_{j=1}^{n} \frac{1}{(\beta + x_j{}^{\alpha})^3} \tag{4.28}$$

Again differentiating l_{33} with respect to γ

$$l_{333} = \frac{\partial^3 L}{\partial \alpha^3} = \frac{2n}{\gamma^3} \tag{4.29}$$

Again differentiating l_{11} with respect to β

$$l_{112} = -(\gamma + 1)\ \omega_{123} \qquad \text{where} \qquad \omega_{123} = \sum_{j=1}^{n} \frac{x_j{}^{\alpha}(\log x_j)^2(\beta - x_j{}^{\alpha})}{(\beta + x_j{}^{\alpha})^3} \tag{4.30}$$

$$l_{112} = l_{121} (\because l_{12} = l_{21})$$

Again differentiating l_{11} with respect to β

$$l_{113} = -\beta \sum_{j=1}^{n} \frac{(\log x_j)^2 x_j{}^{\alpha}}{(\beta + x_j{}^{\alpha})^2} = -\beta\ \omega_{122} \tag{4.31}$$

$$l_{113} = l_{131}$$

Again differentiating l_{22} with respect to α

$$l_{221} = -2(\gamma + 1)\omega_{113} \qquad \text{where} \qquad \omega_{113} = \sum_{j=1}^{n} \frac{x_j{}^{\alpha} \log x_j}{(\beta + x_j{}^{\alpha})^3} \tag{4.32}$$

$$l_{221} = l_{212}$$

Again differentiating l_{22} with respect to γ

$$l_{223} = \frac{-n}{\beta^2} + \delta_{12} \tag{4.33}$$

$$l_{223} = l_{232}$$

Again differentiating l_{33} with respect to α

$$l_{331} = 0 = l_{313} \tag{4.34}$$

Again differentiating l_{33} with respect to β

$$l_{332} = 0 = l_{323} \tag{4.35}$$

Again differentiating l_{23} with respect to α

$$l_{231} = \frac{\partial}{\partial \beta}\left(\frac{\partial^2 L}{\partial \gamma \partial \alpha}\right) = 0 = l_{213} \tag{4.36}$$

Again differentiating l_{12} with respect to β

$$l_{122} = \frac{\partial}{\partial\alpha}\left(\frac{\partial^2 L}{\partial\beta^2}\right) = -2(\gamma+1)\omega_{113} \tag{4.37}$$

Again differentiating l_{13} with respect to β

$$l_{132} = \frac{\partial}{\partial\alpha}\left(\frac{\partial^2 L}{\partial\gamma\partial\beta}\right) = \sum_{j=1}^{n}\frac{x_j{}^{\alpha}\log x_j}{(\beta+x_j{}^{\alpha})^2} = \omega_{112} \tag{4.38}$$

$$l_{132} = l_{123}$$

Again differentiating l_{13} with respect to γ

$$l_{133} = \frac{\partial}{\partial\alpha}\left(\frac{\partial^2 L}{\partial\gamma^2}\right) = 0 \tag{4.39}$$

Again differentiating l_{23} with respect to γ

$$l_{233} = \frac{\partial}{\partial\beta}\left(\frac{\partial^2 L}{\partial\gamma^2}\right) = 0 \tag{4.40}$$

The matrix of derivatives is given as

$$[-l_{ijk}] = -\begin{bmatrix} l_{111} & l_{112} & l_{113} \\ l_{221} & l_{222} & l_{223} \\ l_{331} & l_{332} & l_{333} \end{bmatrix}$$

$$[-l_{ijk}] = \begin{bmatrix} \frac{2n}{\alpha^3} + (\gamma+1)\beta\omega_{133} & -(\gamma+1)\omega_{123} & -\beta\omega_{122} \\ -2(\gamma+1)\omega_{113} & \frac{2n\gamma}{\beta^3} - 2(\gamma+1)\delta_{13} & -\frac{n}{\beta^2} + \delta_{12} \\ 0 & 0 & \frac{2n}{\gamma^3} \end{bmatrix}$$

$$[-l_{ijk}] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$Determinant\ of - [l_{ijk}]$

$$D = \{M_{11}[M_{22}M_{33} - 0] + M_{12}[M_{21}M_{33} - 0] - 0\}$$

$$D = -\{M_{11}M_{22}M_{33} + M_{12}M_{21}M_{33}\}$$

$$D = -M_{33}\{M_{11}M_{22} - M_{12}M_{21}\}$$

$Adjoint\ of\ Matrix - [l_{ijk}]$

$Cofactors\ of\ Matrix - [l_{ijk}]$

$$a_{11} = -[M_{22}M_{33} - 0] = -M_{22}M_{33}$$

$$a_{12} = M_{21}M_{33}$$

$$a_{13} = 0$$

$$a_{21} = M_{12}M_{33}$$

$a_{22} = -M_{11}M_{33}$

$a_{23} = M_{11}M_{32} - M_{31}M_{12} = 0$

$a_{31} = M_{22}M_{13} - M_{12}M_{23}$

$a_{32} = M_{11}M_{23} - M_{21}M_{13}$

$a_{33} = M_{11}M_{22} - M_{12}M_{21}$

$$[-l_{ijk}]^{-1} = \frac{(Adjoint\ of[-l_{ijk}])'}{[-l_{ijk}]}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} -\frac{M_{22}M_{33}}{D} & \frac{M_{12}M_{33}}{D} & \frac{M_{22}M_{13} - M_{12}M_{23}}{D} \\ \frac{M_{21}M_{33}}{D} & -\frac{M_{11}M_{33}}{D} & \frac{M_{11}M_{23} - M_{21}M_{13}}{D} \\ 0 & 0 & \frac{M_{11}M_{22} - M_{12}M_{21}}{D} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} Y_{11/D} & Y_{12/D} & Y_{13/D} \\ Y_{21/D} & Y_{22/D} & Y_{23/D} \\ 0 & 0 & Y_{33/D} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Approximate Bayes Estimator

$$\cup\,(\alpha,\beta,\gamma) = \cup \qquad (4.41)$$

$$\hat{\cup}_{AB} = E(\cup\,|\underline{x})$$

evaluated from equation number (4.6) and from joint prior density equation (4.1)

$$G(\alpha,\beta,\gamma) = g(\alpha)g_2(\beta)g_3(\gamma|\beta)$$

$$= \frac{c}{\delta\Gamma\xi}\beta^{-\xi}\gamma^{\xi-1}exp\left[-\left(\frac{\gamma}{\beta}+\frac{\beta}{\delta}\right)\right]$$

$$\rho = \log G \log C - log\delta - log\Gamma\xi + (\xi-1)log\gamma - \xi log\beta$$

$$-\left(\frac{\gamma}{\beta}+\frac{\beta}{\delta}\right) \qquad (4.42)$$

$$= constant - \xi log\beta + (\xi-1)log\gamma - \frac{\gamma}{\beta} - \frac{\beta}{\delta} \qquad (4.43)$$

$$\rho_1 = \frac{\delta\rho}{\delta\beta} = 0 \qquad (4.44)$$

$$\rho_1 = \frac{\delta\rho}{\delta\beta} = \frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta} \qquad (4.45)$$

$$\rho_3 = \frac{\delta\rho}{\delta\gamma} = \frac{\xi-1}{\gamma} - \frac{1}{\beta} \qquad (4.46)$$

values of A, B and P from equation (4.14) to equation (4.16) and from equation (4.17) to equation (4.40)

$A=[\sigma_{11}l_{111}+2\sigma_{12}l_{121}+2\sigma_{13}l_{131}+2\sigma_{23}l_{231}+\sigma_{22}l_{221}+\sigma_{33}l_{331}]$

$A=\frac{Y_{11}}{D}\left(\frac{2n}{\alpha^3}-(\gamma+1)\omega_{133}\right)+\frac{2Y_{12}}{D}(-(\gamma+1)\omega_{123})+\frac{2Y_{13}}{D}(-\beta\omega_{122})+\frac{2Y_{23}}{D}.O+\frac{Y_{22}}{D}(-2(\gamma+1)\omega_{113})+\frac{Y_{33}}{D}.O$

$$A=\frac{1}{D}\left[Y_{11}\left(\frac{2n}{\alpha^3}-(\gamma+1)\omega_{133}\right)-2Y_{12}(\gamma+1)\omega_{123}-2Y_{13}\beta\omega_{122}-2Y_{22}(\gamma+1)\omega_{113}\right] \quad (4.47)$$

$B=[\sigma_{11}l_{112}+2\sigma_{12}l_{122}+2\sigma_{13}l_{132}+2\sigma_{23}l_{232}+\sigma_{22}l_{222}+\sigma_{33}l_{332}]$

$B=-\frac{Y_{11}}{D}(\gamma+1)\omega_{123}+2\frac{Y_{12}}{D}(-2(\gamma+1)\omega_{113})+\frac{2Y_{13}}{D}\omega_{112}+\frac{2Y_{23}}{D}\left(-\frac{n}{\beta^2}+\delta_{12}\right)+\frac{Y_{22}}{D}\left(\frac{2n\gamma}{\beta^3}-2(\gamma+1)\delta_{13}\right)+\frac{Y_{33}}{D}.O$

$$=\frac{1}{D}\left[-Y_{11}(\gamma+1)\omega_{123}-4Y_{12}(\gamma+1)\omega_{113}+2Y_{13}\omega_{112}+2Y_{23}\left(\frac{-n}{\beta^2}+\delta_{12}\right)+Y_{22}\left(\frac{2n\gamma}{\beta^3}-2(\gamma+1)\delta_{13}\right)\right] \quad (4.48)$$

$P=[\sigma_{11}l_{113}+2\sigma_{12}l_{123}+2\sigma_{13}l_{133}+2\sigma_{23}l_{233}+\sigma_{22}l_{223}+\sigma_{33}l_{333}]$

$=-\frac{Y_{11}}{D}\beta\omega_{122}+\frac{2Y_{12}}{D}\omega_{112}+\frac{2Y_{13}}{D}.O+\frac{2Y_{23}}{D}.O+\frac{Y_{22}}{D}\left(\frac{-n}{\beta^2}+\delta_{12}\right)+\frac{Y_{33}}{D}\frac{2n}{\gamma^3}$

$$=\frac{1}{D}\left[2Y_{12}\omega_{112}-Y_{11}\beta\omega_{122}+2Y_{33}\frac{n}{\gamma^3}+\frac{Y_{22}}{D}\left(\frac{-n}{\beta^2}+\delta_{12}\right)\right] \quad (4.49)$$

$\hat{U}_{AB}=E(U\,|\underline{x})$

$$=u+(u_1a_1+u_2a_2+u_3a_3+a_4+a_5)+\frac{1}{2}[A(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13})+B(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23})+P(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33})]+0\left(\frac{1}{n^2}\right)$$

$$E(U\,|\underline{x})=U+\varphi_1+\varphi_2 \quad (4.50)$$

Where

$\varphi_1=u_1a_1+u_2a_2+u_3a_3+a_4+a_5$

$\varphi_2=\frac{1}{2}[(A\sigma_{11}+B\sigma_{21}+P\sigma_{31}).U_1+(A\sigma_{12}+B\sigma_{22}+P\sigma_{32}).U_2+(A\sigma_{13}+B\sigma_{23}+P\sigma_{33})U_3]$

evaluated at the MLE $\hat{U}=(\hat{\alpha},\hat{\beta},\hat{\gamma})$ where

$a_1=\rho_1\sigma_{11}+\rho_2\sigma_{12}+\rho_3\sigma_{13}$

$$=0.\sigma_{11}+\left(\frac{-\xi}{\beta}+\frac{\gamma}{\beta^2}-\frac{1}{\delta}\right)\frac{Y_{12}}{D}+\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)\frac{Y_{13}}{D} \quad (4.51)$$

$a_2=\rho_1\sigma_{21}+\rho_2\sigma_{22}+\rho_3\sigma_{23}$

$$=0.\sigma_{21}+\left(\frac{-\xi}{\beta}+\frac{\gamma}{\beta^2}-\frac{1}{\delta}\right)\frac{Y_{22}}{D}+\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)\frac{Y_{23}}{D} \quad (4.52)$$

$a_3=\rho_1\sigma_{31}+\rho_2\sigma_{32}+\rho_3\sigma_{33}$

$$= 0.\sigma_{31} + \rho_2.0 + \left(\frac{\xi-1}{\gamma} - \frac{1}{\beta}\right)\frac{Y_{33}}{D} \quad (4.53)$$

$$a_4 = U_{12}\sigma_{12} + U_{13}\sigma_{13} + U_{23}\sigma_{23}$$

$$= \frac{1}{D}(Y_{12}U_{12} + Y_{13}U_{13} + Y_{23}U_{23}) \quad (4.54)$$

$$a_5 = \frac{1}{2}(U_{11}\sigma_{11} + U_{22}\sigma_{22} + U_{33}\sigma_{33})$$

$$= \frac{1}{2D}(Y_{11}U_{11} + Y_{22}U_{22} + Y_{33}U_{33}) \quad (4.55)$$

Approximate Bayes Estimator under Linex Loss function

$$\hat{U}_{ABL} = -\frac{1}{k}log(E_u(e^{-ku}))$$

where

$$E_u(e^{-ku}|\underline{x}) = \frac{\int_\alpha\int_\beta\int_\gamma e^{-ku}G^*(\alpha,\beta,\gamma)\partial\alpha\partial\beta\partial\gamma}{\int_\alpha\int_\beta\int_\gamma G^*(\alpha,\beta,\gamma)\partial\alpha\partial\beta\partial\gamma} \quad (4.56)$$

The above equation (4.56) is evaluated by method of Lindley approximation, whose simplified form is equation(4.50)

Special Cases :–

$U(\alpha,\beta,\gamma) = U$

1. Approximate Bayes Estimate of γ

$U(\alpha,\beta,\gamma) = U = e^{-k\gamma}$

$U_1 = U_{12} = U_{13} = U_{11} = 0$

$U_2 = U_{21} = U_{22} = U_{23} = 0$

$$U_3 = \frac{\partial}{\partial\gamma}(e^{-k\gamma}) = -ke^{-k\gamma}; \quad U_{33} = \frac{\partial}{\partial\gamma}(-e^{-k\gamma}) = k^2e^{-k\gamma}, \quad U_{31} = U_{32} = 0$$

$$\varphi_1 = U_1\, a_1 + U_2\, a_2 + U_3\, a_3 + a_4 + a_5$$

$$= 0.a_1 + 0.a_2 - e^{-k\gamma}\left(\frac{\xi-1}{\gamma} - \frac{1}{\beta}\right)\frac{Y_{33}}{D} + 0 + \frac{1}{2D}k^2e^{-k\gamma}$$

$$= -ke^{-k\gamma}\left(\left(\frac{\xi-1}{\gamma} - \frac{1}{\beta} - \frac{k^2}{2}\right)\frac{Y_{33}}{D}\right)$$

$$\varphi_2 = \frac{1}{2}[(A\sigma_{11} + B\sigma_{21} + P\sigma_{31})\, U_1 + (A\sigma_{12} + B\sigma_{22} + P\sigma_{32})\, U_2 + (A\sigma_{13} + B\sigma_{23} + P\sigma_{33})\, U_3]$$

$$= -ke^{-k\gamma}\left(\frac{A\sigma_{13} + B\sigma_{23} + P\sigma_{33}}{2}\right)$$

$$E(u|\underline{x}) = e^{-k\gamma} - ke^{-kr}\left(\frac{\xi-1}{\gamma} + \frac{\gamma}{\beta} - \frac{k^2}{2}\right)\frac{Y_{33}}{D} - ke^{-kr}\left(\frac{A\sigma_{13} + \sigma_{23}B + P\sigma_{33}}{2}\right)$$

$$E(u|\underline{x}) = e^{-k\gamma}\left[1 - k\left(\frac{\xi-1}{\gamma}+\frac{1}{\beta}-\frac{k^2}{2}\right)\frac{Y_{33}}{D}+\left(\frac{A\sigma_{13}+B\sigma_{23}+P\sigma_{33}}{2}\right)\right]$$

$$E(u|\underline{x}) = e^{-k\gamma}\varphi_5$$

$$\hat{\gamma}_{ABL} = \gamma + k' log\varphi_5; \text{ at } (\hat{\alpha}_{ML,}\hat{\beta}_{ML,}\hat{\gamma}_{ML}) \tag{4.57}$$

Where

$$\varphi_5 = 1 - k\left(\frac{\xi-1}{\gamma}+\frac{1}{\beta}-\frac{k^2}{2}\right)\frac{Y_{33}}{D}+\left(\frac{A\sigma_{13}+B\sigma_{23}+P\sigma_{33}}{2}\right)$$

Simulation and Numerical Comparison

The simulations and numerical calculations are done by using R Language programming and results are presented in form of table(1).

1. The Random variable of Generalized Compound Rayleigh Distribution is generated by R-Language programming by taking the values of the parameters α, β, γ , taken as $\alpha = 0.9$, $\beta = 0.6$ and $\gamma = 0.5$ in the equations[(4.2)-(4.4)] and equation(1.1).

2. Taking the different sizes of samples n=10(10)80 with complete sample, MLE's, the Approximate Bayes estimator, and their respective MSE's (in parenthesis) are obtained by repeating the steps 500 times, are presented in the tables from (1), for t=0.4, k= -8, R(t)=0.34, H(t)=0.45 and parameters of prior distribution a =2.5 and b =3.4.

3. Table (1) presents the MLE of α, β and γ and Approximate Bayes estimators of γ (for α, β and γ unknown) under LLF. The MSE's in all above cases are presented in parenthesis.

Table (1)
Mean and MSE'S of α, β, γ
($\alpha = 0.9$, $\beta = 0.6$ and $\gamma = 0.5$)

n	$\hat{\alpha}_{ML}$	$\hat{\beta}_{ML}$	$\hat{\gamma}_{ML}$	$\hat{\gamma}_{BL}$	$\hat{\gamma}_{ABL}$
10	0.648415	0.66795	0.60056	0.61431	0. 85476
	[0.891112]	**[0.049521]**	**[0.032419]**	**[1.223x10^{-3}]**	**[3.444x10^{-2}]**
20	0.77054	0.49999	0.67545	0.719821	0. 77564
	[0.98342]	**[0.84537]**	**[0.035421]**	**[1.5994x10^{-2}]**	**[3.1912x10^{-2}]**
30	0.7751648	0.558288	0.6865478	0. 739252	0. 793433
	[0.057125]	**[0.658458]**	**[0.096548]**	**[.02589461]**	**[0.4317145]**
40	0.79865421	0.5482658	0.86975682	0. 768581	0. 8937546
	[0.46572]	**[0.003254]**	**[0.003272]**	**[0.0074623]**	**[0.85739]**
50	0.8973214	0.603484	0.839511	0.886543	0.865738
	[0.004578]	**[0.004577]**	**[0.004265]**	**[0.001624]**	**[0.015437]**
60	0.91478523	0.6990011	0.9490011	0.9757613	0.9792358

	[0.004325]	**[0.054663]**	**[0.004226]**	**[0.001624]**	**[0.015437]**
70	1.000001	0.6988845	0.9454543	0.9924443	0.9807398
	[0.000125]	**[0.001125]**	**[0.001367]**	**[0.003718]**	**[0.010374]**
80	1.2354782	0.7235814	0.9657432	1.0524443	1.007398
	[0.325874]	**[0.025258]**	**[0.001245]**	**[0.004012]**	**[0.010544]**

References:

1. Aitchison, J. and Dunsmore, I.R. (1975): *"Statistical Prediction Analysis"*, Cambridge University Press, London.
2. Basu, A.P. and Ebrahimi, N. (1991): *"Bayesian Approach to Life Testing and Reliability Estimation Using Asymmetric Loss Function"*, J Statist, Vol. 15, pp 190-204.
3. Berger, J.O. (1980): *"Statistical Decision Theory Foundations, Concepts and Methods"*, New York; Springer Varlag.
4. Burr, I.W. (1942): *"Cumulative Frequency Distribution"*, Annals of Mathematical Statistics, Vol. 13, pp 215-232.
5. Ferguson, Thomas S. (1967): *"Mathematical Statistics: A Decision Theoretic Approach"*. Academic press, New York.
6. Lindley, D.V. (1980): *"Approximate Bayesian Method"*. Trab. Esta., Vol. 31, pp 223-237.
7. Rojo (1987): *"On The Admissibility of Cx+D With respect to the LINEX Loss Function,"* commun. Statist theory Math. Vol. 16, No. 12, Vol. 3745-3748
8. Sinha, S.K. (1986): *"Reliability And Life Testing"*, Wiley Eastern Ltd., Delhi, India.
9. Sinha, S.K. and Sloan, J.A. (1988): *"Bayesian Estimation of Parameters and Reliability Function Of The 3-Paramters Weibull Distrib."*. IEEE Tra. Rel., V-R-37, pp 364-369.
10. Soliman, A.A. (2001): *"Linex and Quadratic Approximate Bayesian Estimators Applied to Pareto Model",* Communication in Statistics - Simulation Vol. 30, No.1, pp 47-62.
11. Varian, H.R. (1975):*"A Bayesian Approach to Real Estate Assessment, in Studies in Bayesian Econometrics and Statistics in Honor of Leonard J.Savage",* S.E.Fienberg and A.Zellner, Eds. pp. 195-208, North-Holland Amsterdam, the Neterlands.
12. Zellner, A. (1986): *"Bayesian Estimation and Prediction Using Asymmetric Loss Function"*, Journal of Amer. Statist. Assoc., Vol. 81, pp. 446–451.
13. Zellner, A. and Giesel, M.S. (1968): *"Sensitivity of Control to Uncertainty and Form of Criterion Function in the Future of Statistics"*. ed. Donal G. Watts. N Y Academic Press.