Hsu – Structure in the Cartesian product Manifold

Dr. Navneet Kumar Verma & Dr. Shavej Ali Siddiqui
Sagar Institute of Technology & Management
Barabanki, Pin-225001, Uttar Pradesh, India

Abstract: - Cartesian product of two manifolds has been defined and studied by Petrov [1] and others. In this paper, we have taken Cartesian product of p manifolds, where p is a positive integer. Certain prospects of this product manifold have been defined and studied. The curvature tensor of the product manifold has also been studied and it has been shown that the Cartesian product manifold is an Einstein space if the constituent manifolds also possess the same property.

Keywords:- Differentiable manifolds, Hsu-structure, Hsu-Kahler manifold, curvature tensor, Ricci tensor, Einstein space

Introduction:-

Let $M_1, M_2, \ldots, M_p$ be p differentiable manifolds each of class C^∞. Let $n_1, n_2, \ldots, n_p$ be dimensions of these manifolds. Suppose $m_1, m_2, \ldots, m_p$ be points of manifolds $M_1, M_2, \ldots, M_p$ respectively and let $(M_1)m_1, (M_2)m_2, \ldots, (M_p)m_p$ be their tangent spaces at these points.

Consider the Cartesian product manifold $M_1 \times M_2 \times \ldots \times M_p$. If $X_1, X_2, \ldots X_p$ be tangent vectors of $M_1, M_2, \ldots, M_p$ respectively, then $(X_1, X_2, \ldots X_p)$ is a tangent vector of the product manifold $M_1 \times M_2 \times \ldots \times M_p$.

Let us define vector addition and scalar multiplication on the Cartesian product manifold as follows

(1.1.1) $(X_1, X_2, \ldots, X_p) + (Y_1, Y_2, \ldots, Y_p) = (X_1 + Y_1, X_2 + Y_2, \ldots, X_p + Y_p)$

and

(1.1.2) $\lambda(X_1, X_2, \ldots, X_p) = (\lambda X_1, \lambda X_2, \ldots, \lambda X_p)$

Where X_i, Y_i are tangent vectors of the manifold M_i, $1 \leq i \leq p$ and λ a scalar. Then it is easy to show that the set of all tangent vectors of the product manifold forms a vector space under the vector addition and scalar multiplication defined above.

Define (1, 1) tensor field F for the product manifold as follows:

(1.1.3) $F(X_1, X_2, \ldots, X_p) = (F_1X_1, F_2X_2, \ldots, F_pX_p)$

Where $F_1, F_2, \ldots, F_p$ are (1, 1) tensor fields on $M_1, M_2, \ldots, M_p$ respectively.

If $f_1, f_2, \ldots, f_p$ are C^∞ functions on $M_1, M_2, \ldots, M_p$ respectively, then $(f_1, f_2, \ldots, f_p)$ is the C^∞ function on the product manifold defined as

(1.1.4) $$(X_1, X_2, \ldots, X_p)(f_1, f_2, \ldots, f_p) = (X_1 f_1, X_2 f_2, \ldots, X_p f_p)$$

For connections $D_1, D_2, \ldots, D_p$ on manifolds $M_1, M_2, \ldots, M_p$, let us define the connection D on the product manifold as follows :

(1.1.5) $$D_{(X_1, X_2, \ldots, X_p)}(Y_1, Y_2, \ldots, Y_p) = (D_{1_{X_1}} Y_1, D_{2_{X_2}} Y_2, \ldots, D_{p_{X_p}} Y_p)$$

It is easy to show that D satisfies all the properties of connection on the product manifold.

A manifold is said to possess Hsu-structure if it admits a tensor field ϕ of type (1, 1) satisfying

(1.1.6) $$\phi^2 = a^r I$$

Where 'a' is a real or complex number and 'r' a positive integer

1.2 Some Results

In this section, we shall prove the following theorems:

Theorem (1.2.1) The product manifold $M_1 \times M_2 \times \ldots \times M_p$ will admit Hsu-structure if and only if $M_1, M_2, \ldots, M_p$ are Hsu-structure manifolds.

For if $M_1, M_2, \ldots, M_p$ are Hsu-structure manifolds, we have

$$F_i^2 X_i = a^r X_i, \quad i = 1, 2, \ldots, p.$$

Now

$$\begin{aligned} F^2(X_1, X_2, \ldots, X_p) &= (F_1^2 X_1, F_2^2 X_2, \ldots, F_p^2 X_p) \\ &= (a^r X_1, a^r X_2, \ldots, a^r X_p) \\ &= a^r (X_1, X_2, \ldots, X_p) \end{aligned}$$

or

(1.2.1) $F^2(X_1, X_2, \ldots, X_p) = a^r(X_1, X_2, \ldots, X_p)$

Hence the proposition Converse can be proved easily.

Let us now define the Riemannian metric 'g' on the product manifold $M_1 \times M_2 \times \ldots \times M_p$ as follows:

(1.2.2) $g((X_1, X_2, \ldots, X_p), (Y_1, Y_2, \ldots, Y_p)) = g_1(X_1, Y_1) + g_2(X_2, Y_2) + \ldots + g_p(X_p, Y_p)$

Where $g_1, g_2, \ldots, g_p$ are Riemannian metrics on manifolds $M_1, M_2, \ldots, M_p$ respectively.

Let $\xi_1, \xi_2, \ldots, \xi_p$ be vector fields and $\eta_1, \eta_2, \ldots, \eta_p$ 1-forms on manifolds $M_1, M_2, \ldots, M_p$ respectively. We define a vector field ξ and a 1-form η on the product manifold as follows :

(1.2.3) $\eta (X_1, X_2, \ldots, X_p)\xi = (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \ldots, \eta_p(X_p)\xi_p)$

Where $X_1, X_2, \ldots, X_p$ are vector fields on the manifolds $M_1, M_2, \ldots, M_p$ respectively.

Now, we have

Theorem (1.2.2) The product manifold $M_1 \times M_2 \times \ldots \times M_p$ admits Hsu-contact structure if the constituent manifolds $M_1, M_2, \ldots, M_p$ possess the same structure.

Proof: Since the constituent manifolds $M_1, M_2, \ldots, M_p$ admit Hsu-contact structure hence

(1.2.4) $F_i^2 X_i = a^r X_i + n_i(X_i)\xi_i$

$i = 1, 2, \ldots, p$. Now

$F^2(X_1, X_2, \ldots, X_p) = (F_1^2 X_1, F_2^2 X_2, \ldots, F_p^2 X_p)$

So equation (1.2.4) takes form

$$F^2(X_1, X_2, \ldots, X_p) = a^r(X_1, X_2, \ldots, X_p) + (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \ldots, \eta_p(X_p)\xi_p)$$

or

$$F^2(X_1, X_2, \ldots, X_p) = a^r(X_1, X_2, \ldots, X_p) + \eta(X_1, X_2, \ldots, X_p)\,\xi$$

Hence the Cartesian product manifold $M_1 \times M_2 \times \ldots \times M_p$ admits the Hsu-contact structure.

Theorem (1.2.3) The Cartesian product manifold $M_1 \times M_2 \times \ldots \times M_p$ is Hsu-Kahler manifold iff the constituent manifolds also have the same structure.

The product manifold $M_1 \times M_2 \times \ldots \times M_p$ will be Hsu-Kahler iff $(D_X F)\,Y = 0$ for arbitrary vector field X, Y on the product manifold.

Taking $X = (X_1, X_2, \ldots, X_p)$, $Y = (Y_1, Y_2, \ldots, Y_p)$. Then

$$(D_X F)Y = (D_{(X_1,X_2,\ldots,X_p)}F)(Y_1, Y_2, \ldots, Y_p)$$

$$= D_{(X_1,X_2,\ldots,X_p)}F(Y_1,Y_2,\ldots,Y_p) - F(D_{(X_1,X_2,\ldots,X_p)}(Y_1,Y_2,\ldots,Y_p))$$

$$= D_{(X_1,X_2,\ldots,X_p)}(F_1Y_1,F_2Y_2,\ldots,F_pY_p) - (F_1D_{1_{X_1}}Y_1,F_2D_{2_{X_2}}Y_2,\ldots,F_pD_{p_{X_p}}Y_p)$$

(1.2.5) $$= ((D_{1_{X_1}}F_1)Y_1,(D_{2_{X_2}}F_2)Y_2,\ldots,(D_{p_{X_p}}F_p)Y_p) = 0$$

It is possible only when $(D_{1_{X_1}}F_1)Y_1 = 0$, $(D_{2_{X_2}}F_2)Y_2 = 0$, …, $(D_{p_{X_p}}F_p)Y_p = 0$.

Hence the product manifold will be Hsu-Kahler if $M_1, M_2, \ldots, M_p$ are of same type and vice versa.

Theorem (1.2.4) The Cartesian product manifold $M_1 \times M_2 \times \ldots \times M_p$ will be Hsu nearly Kahler iff the constituent manifold have the similar structure.

Proof: The product manifold $M_1 \times M_2 \times \ldots \times M_p$ will be Hsu nearly Kahler iff

$$(D_XF)Y + (D_YF)X = 0$$

As in the previous theorem, we can write above equation as

(1.2.6) $$(D_XF)Y + (D_YF)X =$$

$$((D_{1_{X_1}}F_1)Y_1 + (D_{1_{Y_1}}F_1)X_1,(D_{2_{X_2}}F_2)Y_2 + (D_{2_{Y_2}}F_2)X_2,\ldots,(D_{p_{X_p}}F_p)Y_p + (D_{p_{Y_p}}F_p)X_p)$$

Hence in order that $(D_XF)Y + (D_YF)X = 0$, it is necessary and sufficient that

$$(D_{i_{X_i}}F_i)Y_i + (D_{i_{Y_i}}F_i)X_i = 0,\ i = 1, 2, \ldots, p.$$

Hence the proposition

1.3 Curvature of the Cartesian product manifold

Let $X = (X_1, X_2, \ldots, X_p)$, $Y = (Y_1, Y_2, \ldots, Y_p)$ and be two C^∞ vector fields on the Cartesian product manifold

$M_1 \times M_2 \times \dots \times M_p$. If $f = (f_1, f_2, \dots, f_p)$ be a C^∞ function on the product manifold, then by definition of the Lie bracket

$$[(X_1, X_2, \dots, X_p), (Y_1, Y_2, \dots, Y_p)]\,(f_1, f_2, \dots, f_p)$$

$$= (X_1, X_2, \dots, X_p)(Y_1 f_1, Y_2 f_2, \dots, Y_p f_p) - (Y_1, Y_2, \dots, Y_p)(X_1 f_1, X_2 f_2, \dots, X_p f_p)$$

or

$$[(X_1, X_2, \dots, X_p), (Y_1, Y_2, \dots, Y_p)]\,(f_1, f_2, \dots, f_p)$$

(1.3.1) $$= ([X_1, Y_1]f_1, [X_2, Y_2]f_2, \dots, [X_p, Y_p]f_p)$$

Suppose $K_i(X_i, Y_i, Z_i)$, $i = 1, 2, \dots, p$ be curvature tensors of the constituent manifolds $M_1, M_2, \dots, M_p$. If $K(X, Y, Z)$ be the curvature tensor of the Cartesian product manifold $M_1 \times M_2 \times \dots \times M_p$, we have

(1.3.2) $$K(X, Y, Z) = (K_1(X_1, Y_1, Z_1), K_2(X_2, Y_2, Z_2), \dots, K_p(X_p, Y_p, Z_p))$$

If $W = (W_1, W_2, \dots, W_p)$ be another vector field on the product manifold, let us put

$$'K(X, Y, Z, W) = g(K(X, Y, Z), W)$$

Then it can be easily shown that

(1.3.3) $$'K(X, Y, Z, W) = {'K_1}(X_1, Y_1, Z_1, W_1) + {'K_2}(X_2, Y_2, Z_2, W_2)$$

$$+ {'K_3}(X_3, Y_3, Z_3, W_3) + \dots + {'K_p}(X_p, Y_p, Z_p, W_p)$$

Hence if the manifold $M_1, M_2, \dots, M_p$ are of constant curvature, the product manifold is also of constant curvature.

In view of the equation (5.3.3), it follows that the Ricci tensor of the product manifold is the sum of the Ricci tensors of the manifolds $M_1, M_2, \dots, M_p$.

If the product space is an Einstein space, we have

(1.3.4) $S(X, Y) = \lambda\, g(X, Y)$, X, Y vector fields and $S(X, Y)$ the Ricci tensor of the product manifold.

In view of the equations (5.3.3) and (5.3.4), it follows that

(1.3.5) $S_i(X_i, Y_i) = \lambda\, g_i(X_i, Y_i)$, $i = 1, 2, \dots, p$.

Hence the manifolds M_1, M_2, …, M_p are also Einstein manifolds. Thus if the product manifold is Einstein manifold, the same are constituent manifolds M_1, M_2, …, M_p.

Also

(1.3.6) $$\lambda = \frac{R}{n} = \frac{R_1}{n_1} = \frac{R_2}{n_2} = \ldots = \frac{R_p}{n_p}$$

Where R, R_1, R_2, …, R_p are scalar curvatures of the product manifold and constituent manifolds and n, n_1, n_2, …, n_p their respective dimensions. Further

(1.3.7) $S = S_1 + S_2 + \ldots + S_p$ provided that the manifolds are Einstein manifolds.

References

1. A.Z. Petrov (1969): Einstein Spaces Pergamon Press, Oxford.
2. Hicks, N.J. (1965): Notes on Differential Geometry D. Van Nostrand Company Inc., New York.
3. K. Yano (1965): Differential geometry on Complex and Almost Complex Spaces Pergmon Press.
4. K. Yano and M. Kon (1984) : Structures on Manifolds World Scientific Publishing Company Pvt. Ltd., P.O. Box 128, Farrer Road, Singapore, 9128.
5. Lovejoy S. Das and Ram Nivas: Harmonic morphism an almost r-contact metric manifolds Algebras Groups and Geometries, 22 (2005), 61-68.
6. Lovejoy S. Das and Ram Nivas: On differentiable manifold with $[F_1, F_2]$ (K+1, 1) – structure Tensor, N.S., 65 (2004), Japan, 29-35.
7. Lovejoy S. Das and Ram Nivas: On certain structures defined on the tangent bundle Rocky Mountain J. of Mathematics, U.S.A., vol. 36, no. 6 (2006), pp. 1857-1866.
8. R.S. Mishra (1984): Structures on a differentiable manifold and their applications Chandrama Prakashan, 50A Balrampur House, Allahabad, India.