SOLUTION OF THE DIOPHANTINE EQUATION

DR. JAI NANDAN SINGH
Head, Dept of Mathematics, Koshi college Khagaria,
T.M. Bhagalpur University Bhagalpur

ABSTRACT :- The Diophantine equation

$$-2 + x + x^3 + 2x^4 = y^2$$

has no integer solution

Keywords :-
Equation, Divisor, integer, Diophantine

Introduction :-

Diophantine equations are polynomial equations where all the variables are integers, and solving them for integer solutions can be quite complex. While some specific Diophantine equations have elementary proofs (meaning proofs that rely only on basic arithmetic), the field in general requires more advanced techniques. Mathematics thrives on equations. relationships between variables that unlock a universe of possibilities. Yet, some problems demand a more specific kind of solution: whole numbers, the integers that form the backbone of counting and discrete quantities. This is the domain of Diophantine equations, named after the Hellenistic mathematician Diophantus, and within this realm, linear Diophantine equations hold a place of fundamental importance. A linear Diophantine equation is a linear equation where the unknowns and the constants are all integers. We typically express them in the from ax + by = c, where a, b, and c are integers, and x and y are the integer unknowns we seek. The beauty of linear Diophantine equations lies in their relative simplicity and the rich tapestry of applications they offer. Determining if solutions exist and finding them often hinges on the greatest common divisor (GCD) of a and b. If the GCD does not divide c, the equation has no integer solutions. However, if the GCD divides c, then there exist infinitely many integer solutions. This paves the way for elegant algorithms to find particular solutions and express all solutions in a general form.

There are three types of Diophantine Equations :-

a) The equation which have no solutions

b) The equation which have only infinitely many solutions.

c) The equation which have only finitely many solutions.

The above Diophantine Equation

$$-2 + x + x^3 + 2x^4 = y^2$$

is of the type (a). Congruence methods provide a useful tool in determining the number of solution to this diophantine equation.

$$-2 + x + x^3 + 2x^4 = y^2$$

Proof :-

The Diophantine equation

$$-2 + x + x^3 + 2x^4 = y^2 \qquad (1)$$

has no integer solutions

The equation (1) can be written as

$$2x^4 + x^3 - 2x^2 - 2x^2 + x - 2 = y^2$$

$$\Rightarrow x^2(2x^2 + x - 2) + 1(2x^2 + x - 2) = y^2$$

$$(x^2 + 1)(2x^2 + x - 2) = y^2 \qquad (2)$$

Let D= $(x^2 + 1, 2x^2 + x - 2)$. Then D divides

$$2x^2 + x - 2 - 2(x^2 + 1) = x - 4$$

also divides $(x^2 + 1) - x(x - 4) = 4x + 1$

Therefore D divides $\begin{vmatrix} 1 & -4 \\ 4 & 1 \end{vmatrix} = 1+16=17$

Hence D=1 or 17

Therefore from equation (2) we have

$$\begin{aligned} x^2+1 &= 17^j m^2 \\ 2x^2+x-2 &= 17^j n^2 \end{aligned} \qquad (3)$$

Where $(m,n)=1, m>0, n>0$

If x were odd, then m is even and the first equation in (3) is impossible to mod4, since $x^2 \equiv 1(mod4), m^2 \equiv 0(mod4)$ so

$2 \equiv 0(mod4)$ which is impossible.

Hence x is even and m odd and then n even from the second equation of (3)

Case1. If j=0. Then from the eqution (3) we have

$$\begin{aligned} x^2+1 &= m^2 \\ 2x^2+x-2 &= n^2 \end{aligned} \qquad (4)$$

Where x is even, m is odd and n is even.

From first equation of (4) we have

$x^2+1=m^2$

or $m^2-x^2=1$

or, $(m+x)(m-x)=1$

so $m+x=\pm 1$

and $m-x=\pm 1$

thus $2x=0$

But $2 \neq 0 \Longrightarrow x=0$

Hence $m=1$

So from equation 2nd of (4) $-2=n^2$ which is impossible.

Thus second equation in (3) is impossible.

Case2. For $j=1$, the equation (3) become

$$\begin{aligned} x^2+1 &= 17m^2 \\ 2x^2+x-2 &= 17n^2 \end{aligned} \qquad (5)$$

Where x is even, m is odd and n is even.

After multiplying first equation of (5) by 2 and subtracting from the second equation of (5).

$$x-4=17(n^2-2m^2)$$

$$x=4+17(n^2-2m^2) \qquad (6)$$

Substitutiong this value of x in the first equation of (5) we have

$$\{4+17(n^2-2m^2)\}^2+1=17m^2$$

or, $16+2.4.17(n^2-2m^2)+17^2(n^2-2m^2)^2+1=17m^2$

or, $17+8.17(n^2-2m^2)+17^2(n^2-2m^2)^2=17m^2$

Dividing through out by 17 we get

$$1+8(n^2-2m^2)+17(n^2-2m^2)^2=m^2$$

or, $1+2.1.4(n^2-2m^2)+16(n^2-2m^2)+(n^2-2m^2)^2=m^2$

or, $1+2.1.4(n^2-2m^2)+\{4(n^2-2m^2)\}^2+(n^2-2m^2)^2=m^2$

or, $\{1+4(n^2-2m^2)\}^2+(n^2-2m^2)^2=m^2$ (7)

Where the two terms on the left hand side are relatively prime since

$[1+4(n^2-2m^2), n^2-2m^2]=(1, n^2-2m^2)=1$

This equation (7) is of the Pythagorean type and so we have

$$1+4(n^2-2m^2)=\pm(r^2-s^2)$$
$$n^2-2m^2=2rs$$
$$m=(r^2-s^2)>0 \qquad (8)$$

Where (r,s)=1 and r,s are of opposite parity since m is odd and n is even.
Here $rs \neq 0$, Since if $rs=0$ then

$$n^2-2m^2=0$$

Which is impossible in integer because

$$n^2=2m^2 \text{ imply } (\frac{n}{m})^2=2$$

or $\frac{n}{m}=\sqrt{2}$ an irrational numbers
We now suppose $rs \neq 0$ then form equation (8)

$$1+8rs=\pm(r^2-s^2)$$
$$n^2=2(r^2+s^2)^2+2rs \qquad (9)$$

Since $1+4(n^2-2m^2)=\pm(r^2-s^2)$

$\Rightarrow \quad 1+4.2rs=\pm(r^2-s^2) \quad \because\ n^2-2m^2=2rs$

$\Rightarrow \quad 1+8rs=\pm(r^2-s^2)$

Again Since

$$n^2-2m^2=2rs$$

$\Rightarrow \quad n^2=2m^2+2rs$

$\Rightarrow \quad n^2=2(r^2+s^2)^2+2rs \quad \because m=r^2+s^2$

Where r, s are of opposite parity and n is even.
If r even and s odd then

$$1\equiv\pm(r^2-1)(mod8)$$

so r is divisible by 4
If r is odd and s even
Then s is divisible by 4
Now we have $\quad n^2\equiv 2(mod8)$
Which is impossible
For if $\quad n=2k \quad (k\ odd)$
then $\quad n^2=4k^2$
Now $\quad 4k^2\equiv 2\ (mod8)$
or, $\quad 2k^2\equiv 1\ (mod4)$
so $\quad 2\equiv 1 \quad (mod4)$
Which is impossible
Now if $\quad n=4k$
or $\quad n^2=16k^2$
so $\quad n^2=16k^2\equiv 2\ (mod8)$
hence $0\equiv 2(mod8)$
Which is impossible

Conclusion

Elementary solutions play a crucial role in solving specific Diophantine equations. From the straight forward approach of linear equations to the more intricate methods for Pell's equation, these techniques showcase the power of manipulating integers and their properties. However, it's valuable to recognize the limitations of these methods and appreciate the vast landscape of Diophantine equations that may require more sophisticated approaches. As mathematicians continue to explore this fascinating realm, elementary solutions will remain a cornerstone for understanding and solving specific Diophantine equations.

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