SOME PROPERTIES AND APPLICATION OF LAPLACE TRANSFORMS

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Abstract: Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very simple approach just like the applications of transfer functions to solve ordinary differential equations.

Keywords: Laplace Transform, Integration, Differential Equation.

Introduction: Many physical problems when analyzed assumes the form of an ordinary differential equation subjected to a set of initial conditions or boundary conditions.Such problems are referred to as initial value problems and boundary value problems respectively.

Laplace transforms serves as a very useful tool in solving these problems without actually finding the general solution of the differential equation by various known methods.

Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. It finds very wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing.

Definition: If f(t) is a real valued function defined for all t$\geq$ 0 then the Laplace transform of f(t) is denoted by L[f(t)] is defined by

L[f(t)]=$\int_{t=0}^{\infty} e^{-st} f(t)dt$ provided the integral exists.

Standard Basic Properties of Laplace transforms:

If L[f(t)] =$\bar{f}$(s) then,

1.<u>Linearity property</u>:

L[$af(t) + bg(t)$]=aL[$f(t)$]+bL[$g(t)$] where a and b are constants.

Proof: By definition L[f(t)]=$\int_{t=0}^{\infty} e^{-st} f(t)dt$

Therefore L[$af(t) + bg(t)$]=$\int_{t=0}^{\infty} e^{-st} [af(t) + bg(t)]dt$

$$=a\int_{t=0}^{\infty} e^{-st} f(t)dt + b\int_{t=0}^{\infty} e^{-st} g(t)]dt$$

=aL$[f(t)]$+bL$[g(t)]$

<u>2.Change of scale property</u>: L$[\boldsymbol{f(at)}]$=$\frac{1}{a}\bar{f}(\frac{s}{a})$

Proof: By definition L[f(t)]=$\int_{t=0}^{\infty} e^{-st} f(t)dt$ =$\bar{f}(s)$

$$\mathrm{L}[f(at)] = \int_{t=0}^{\infty} e^{-st} f(at)dt$$

Put at=u,adt=du and when t=0,u=0 and t=∞,u=∞

L$[f(at)] = \int_0^{\infty} e^{-\frac{su}{a}} f(u)\frac{du}{a}$ =$\frac{1}{a}\int_0^{\infty} e^{-\frac{su}{a}} f(u)du$ =$\frac{1}{a}\bar{f}(\frac{s}{a})$.

<u>3.Shifting property: </u> L$[e^{at}$f(t)]=$\bar{f}$(s-a)

Proof: By definition L[f(t)]=$\int_{t=0}^{\infty} e^{-st} f(t)dt$ =$\bar{f}(s)$……….(1)

L$[e^{at}$f(t)]=$\int_{t=0}^{\infty} e^{-st}[e^{at}\mathrm{f(t)}]dt$ =$\int_{t=0}^{\infty} e^{-st+at} f(t)dt$

=$\int_{t=0}^{\infty} e^{-(s-a)t} f(t)dt$ ……(2)

Comparing (1) and(2) we get L$[e^{at}$f(t)]=$\bar{f}$(s-a)

<u>4.Derivative of the transform property:</u>

L$[t^n$f(t)]=$(-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$ where n is a positive integer.

Proof: We establish the result by the principle of mathematical induction.

We have $\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt$

Differentiating w.r.t **s** on both sides we have ,

$$\frac{d}{ds}[\bar{f}(s)] = \int_0^{\infty} \frac{\partial}{\partial s}[e^{-st}]f(t)dt$$

In the R.H.S, we shall apply Leibnitz rule for differentiating under the integral sign.

$$\therefore \frac{d}{ds}[\bar{f}(s)] = \int_0^{\infty} e^{-st}(-t)f(t)dt$$

Or $(-1)\frac{d}{ds}[\bar{f}(s)] = \int_0^{\infty} e^{-st}[tf(t)]dt = L[tf(t)]$....................... (1)

This verifies the result for n=1

Let us assume the result to be true for n=k

i.e., $(-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = L[t^k f(t)]$.................. (2)

$$or \quad (-1)^k \frac{d^k}{ds^k}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^k f(t)]dt$$

Differentiating w.r.t **s** again we get,

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})[t^k f(t)]dt$$

i.e.,$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}(-t)[t^k f(t)]dt$

Multiplying by (-1) we get,

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = \int_0^\infty e^{-st}[t^{k+1} f(t)]dt$$

i.e., $(-1)^k \frac{d^{k+1}}{ds^{k+1}}[\bar{f}(s)] = L[t^{k+1} f(t)]$.................... (3)

Comparing (2) and (3) we conclude that the result is true for n= k+1.

Hence by the principle of mathematical induction the result is true for all positive integral values of n.

Thus $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)] = (-1)^n f^{-(n)}(s)$

5.Integral property:

L$\left[\int_0^t f(t)dt\right]$=$\frac{1}{s}\bar{f}$(s) where s>0

Proof: Let $F(t) = \int_0^t f(t)\,dt \quad and\ hence$

$$F'(t) = f(t) \quad and\ F(0) = 0$$

Now $L[F(t)] = \int_0^\infty F(t)e^{-st}\,dt\ by\ the\ definition.$

Integrating by parts we get,$L[F(t)] = \left[F(t)\frac{e^{-st}}{-s}\right]_{t=0}^\infty - \int_0^\infty \frac{e^{-st}}{-s}F'(t)dt$

$$= (0-0) + \frac{1}{s}\int_0^\infty e^{-st} f(t)dt = \frac{1}{s}\bar{f}(s)$$

$$Thus\ L\left[\int_0^t f(t)dt\right] = \frac{\bar{f}(s)}{s}$$

4. Division by t property:

$L\left[\frac{f(t)}{t}\right]=\int_s^{\infty}\bar{f}(s)ds$

Proof: We have $\bar{f}(s)=\int_0^{\infty}e^{-st}f(t)dt$

$$\therefore \int_0^{\infty}\bar{f}(s)ds=\int_s^{\infty}\left[\int_0^{\infty}e^{-st}f(t)dt\right]ds$$

$$i.e.,=\int_0^{\infty}\int_s^{\infty}e^{-st}f(t)ds\,dt\,,\qquad on\ changing\ the\ order\ of\ integration.$$

$$=\int_0^{\infty}\left[\frac{e^{-st}}{-t}\right]_s^{\infty}f(t)dt=\int_0^{\infty}\left[0-\frac{e^{-st}}{-t}\right]f(t)dt$$

$$i.e.,\int_0^{\infty}\bar{f}(s)ds=\int_0^{\infty}e^{-st}\left[\frac{f(t)}{t}\right]dt=L\left[\frac{f(t)}{t}\right]$$

$$Thus\ L\left[\frac{f(t)}{t}\right]=\int_0^{\infty}\bar{f}(s)ds$$

Examples: Find the Laplace transform of the following functions;

1.$e^{-2t}sinh4t$

Soln: $e^{-2t}\left(\frac{e^{4t}-e^{-4t}}{2}\right)=\frac{4}{(s-2)(s+6)}$

2.tcosat

Soln: L[tcosat]= $-\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)=\frac{s^2-a^2}{[s^2+a^2]^2}$

Applications of Laplace Transformation

1. It is widely used to analyze and design control systems. It helps to convert time-domain signals into frequency-domain signals, making it easier to analyze and design the system's behaviour.
2. It is used to analyze and design electrical circuits. In addition, it helps to solve differential equations related to circuits and determine their stability and transient response.
3. It is used in mechanics to analyze the behaviour of mechanical systems, such as structures' vibrations, the pendulum's motion, and system dynamics.
4. It is used to solve partial differential equations. It transforms differential equations into algebraic equations, which are easier to solve.
5. It is used in probability theory to derive the moment-generating function of a probability distribution. The moment-generating function is used to find moments of a distribution, which are useful in statistical analysis.

Conclusion

The Laplace transformation is a powerful mathematical tool that has proven to be essential in many fields of study. For example, made it an invaluable tool for solving differential equations and analyzing linear time-invariant systems. The real-life applications of the Laplace transformation are vast and diverse, ranging from electrical circuits and control systems to economics and physics.

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