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Generalized Fuzzy Measurable Functions in Topological Measurable Space

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ABSTRACT :

This paper introduces a generalized framework for fuzzy measurable functions in topological measurable spaces. We extended the classical concept of fuzzy measurability by incorporating both measurable and topological structures, allowing a unified treatment of generalized fuzzy mappings. Including closure under finite unions and intersections, composition, and stability under suitable conditions with new definitions of generalized fuzzy measurable functions are proposed together with their fundamental properties. Relationships between generalized fuzzy measurable functions, measurable graphs, level-set measurability are investigated. Conditions are sufficient for the existence of generalized fuzzy selection established in topological measurable spaces. The concept of gradual number-valued measurable functions is introduced and some preserving essential measurability properties.

Keywords: Topological measurable spaces, fuzzy mappings, fuzzy measurable functions, fuzzy selection, gradual number-valued measurable function.

1. Introduction :

Motivated by the application in several areas of applied sciences such as fuzzy optimal, process, control and decision theory Mathematical economics, much effort has been devoted to the generalization of different measure concepts and classical results to the case when outcomes of a random experiment represented by fuzzy sets. The concepts of fuzzy measure [1,10,8,9]. Fuzzy set theory introduced by Zadeh, has become an important framework for modelling uncertainty and vagueness in mathematics and applied sciences. The concept of measurable fuzzy function plays a fundamental role in fuzzy analysis, optimization probability, control theory, and decision making. Classical studies mainly investigate fuzzy measurable functions on measurable spaces.

Topological measurable spaces provide a natural setting in which measurable and topological properties coexist.

In this paper, we introduce a new class of generalized fuzzy measurable functions in topological measurable spaces. We establish their basic fuzzy measurable functions [3,7,14] and fuzzy integrals [2,3,4,5]. The principal contribution of this paper is the established several fundamental properties of generalized fuzzy measurable functions in separately, we prove closure properties under finite unions and intersections, composition theorems, product mapping theorem, inverse image properties, generalized measurable graph theorem and generalized existence selector result.

Introduced a new concept in fuzzy set theory "Gradual numbers". A gradual number cannot be considered as fuzzy set real numbers because the mapping from the unit interval to the real line is not necessarily one to one. In the brief time since their introduction, gradual numbers have been computed on fuzzy intervals, with application to combinatorial fuzzy optimization [6,11] and others [12,13,15,16]. This paper is a continuation of paper [18]. It is expected that these developments will stimulate new investigation in both theoretical and applied fuzzy mathematics.

The main contribution of this paper are summarized as follows:

* Introduction of generalized fuzzy measurable functions on topological measurable spaces.
* Establish new composition, graph measurability and corollary.
* Development new generalized fuzzy measurable function definitions.
* Generalized fuzzy measure properties.
* Discussion of strongly generalized integral α-level sets in topological spaces.

The generalized framework developed in this paper provides a foundation for future research on generalized fuzzy measurable multi-functions, generalized fuzzy integrations, generalized composition theorem on α-level sets, strongly generalized fuzzy measurable functions. It also opens new directions for extending fuzzy measurable analysis to broader classes of uncertainty models beyond classical fuzzy theory.

2. Preliminaries:

Definition 2.1 (Generalized fuzzy set):

Let X be a non-empty set equipped with a topology T and σ algebra Σ.

A generalized fuzzy set on X is a mapping

$$G: X \to [0,1]$$

such that, for every $\alpha \in (0,1]$, the generalized α-level set

$$G_\alpha = \{x \in X : G(X) \geq \alpha\}$$

$$g(x) \subseteq \Sigma \text{ and}$$

each member of $g(X)$ is compatible with topology τ.

The value of G(X) is the generalized degree of membership of the element x in the fuzzy set G.

Definition 2.2 (Generalized topological measurable space):

Let X be a non-empty set. Let τ_G be a generalized topology on X, and let Σ_G be a generalized σ-algebra on X. The ordered triple (X, τ_G, Σ_G) is called a generalized measurable space if the following conditions hold:

I. $\phi, X \in \tau_G \cap \Sigma_G$

II. Every generalized open-set $U \in \tau_G$ is generalized measurable, that is

$$\tau_G \subseteq \Sigma_G .$$

III. The family Σ_G is closed under complements and countable union, namely if A, $\beta_n \in \Sigma_G \ \forall \ n \in N$, then $X \setminus B \in \Sigma_G$ and

$$\bigcup_{n=1}^{\infty} B_n \in \Sigma_G.$$

IV. The generalized topology τ_G is compatible with Σ_G in the sense that the closure and interior of every generalized measurable set remain generalized. When $\tau_G = \gamma$ is an ordinary topology & $\Sigma_G = \Sigma$ is the classical Borel σ-algebra.

Definition 2.3 (Generalized gradual numbers):

Let M = (0,1] be the unit interval. A generalized gradual number on a non-empty set X is a mapping

$$\gamma_G : M \times X \to R$$

such that the following conditions are satisfied:

I. For every fixed $x \in X$, the mapping

$$\alpha \to \gamma_G(\alpha, x)$$

is measurable on M.

II. For every fixed $\alpha \in M$, the mapping

$$x \to \gamma_G(\alpha, x)$$

is measurable with respect to the generalized measurable structure Σ_G

III. If $x \in U$, U is a generalized open set of (X, τ_G), then $\gamma_G(\alpha, x)$ is locally continuous with respect to the generalized topology.

The value $\gamma_G(\alpha, x)$ is called the generalized gradual value of x at level α.

where $x = x_0$ is a singleton set, then

$$\gamma_G(\alpha, x_0) = r(\alpha).$$

where $r : (0,1] \to R$ is an ordinary gradual number.

Definition 2.4 (Generalized Fuzzy Measure):

A mapping

$$\mu_G : \Sigma_G \to [0, \infty) \qquad (\mu = \text{Mue})$$

$$\mu_G(\phi) = 0$$

$$A \subseteq B \Rightarrow \mu_G(A) \le \mu_G(B)$$

It satisfies generalized continuity under increasing sequence.

Definition 2.5 (Generalized Fuzzy Optimization Problem):

A problem of the form

$$\text{Max } x \in X \; E_G(\gamma)$$

$$\text{Min } x \in X \; E_G(\gamma)$$

Definition 2.6 (Generalized Fuzzy Integral):

Let $f : X \to [0,1]$. A_G be a generalized fuzzy set and μ_G (mue) a generalized fuzzy integral. Then

$$\int_x f \, d\mu_G = \int_0^1 \mu_G \; (\{x : f(x) \ge \alpha\}) d_\alpha$$

Definition 2.7 (Generalized Fuzzy measurable Selector):

Let

$\psi_G : X \to F(Y)$

$t : X \to Y$

$t(x) \in \psi_G(x)$

Definition 2.8 (Generalized Fuzzy measurable Graph):

The graph of ψ_G,

$$Gr(\psi_G) = \{(x,y) \in X \times Y : y \in \psi_G(x)\}$$

$$Gr(\psi_G) \in \Sigma_G \otimes B_G$$

Definition 2.9 (Strongly Generalized Fuzzy Measurable Function):

Let (X, τ_G, Σ_G) and (Y, η_G, B_G) be generalized topological measurable spaces. Let

$f_G : X \to Y$ be a fuzzy valued mapping. The mapping f_G is called a strongly generalized fuzzy measurable function if, for every generalized fuzzy measurable set $B_G \in \beta_G$

$$f_G^{-1}(B_G) \in \Sigma_G$$

and for every $\alpha \in (0,1]$, the inverse image of the α-level set

$f_G^{-1}(B_G)_\alpha$ is a generalized measurable set in Σ_G.

The generalized Choquet fuzzy integral of f_G with respect to the generalized fuzzy measure μ_G is defined by

$C_G \int_x f_G dx \mu_G = \int_0^1 \mu_G$ $(\{x \in X : f_G(x) \geq \alpha\})d_\alpha$

Here,

$E_\alpha(f_G) = \{x \in X : f_G(x) \geq \alpha\}$

is called the generalized α-level measurable set of f_G.

The integral is said to exist whenever the mapping

$\alpha \rightarrow \mu_G$ $(E_\alpha$ $(f_G))$ is Lebesgue integral on the interval [0,1].

If $\tau_G = \tau$, $\sum_G = \sum$, $\mu_G = \mu$

reduces to the classical Choquet fuzzy integral.

Definition 2.10: (Generalized α-Level-set):

For every $\alpha \in (0,1]$

G_α $(x) = \{y \in \gamma : \mu_{G(x)}$ $(y) \geq \alpha\}$

This is called the generalized α-level set of G.

Lemma 2.11:

Let i: P$\rightarrow$Q, j : Q$\rightarrow$R be arbitrary mappings. Then for every Fuzzy subset $B \subseteq R$, we have

$(j \text{ o } i)^{-1}(B) = i^{-1}(i^{-1}(B))$

3. Properties of the Generalized Fuzzy Measure:

Let $\mu_G : \sum_G \rightarrow [0, \infty]$ then μ_G (mueG) satisfies:

3.1 Null property: μ_G (ϕ)= 0.

3.2 Monotonicity: If $A \subseteq B$, μ_G $(A \leq H_G$ (B).

3.3 Continuity from Below: If $A_1 \subseteq A_2 \subseteq A_3$ S…then

μ_G $(U_{n=1}^{\infty}$ $A_n) = \lim_\infty \mu_G$ (A_n)

3.4 Continuity from Above: If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and $\mu_G(A_1) < \infty$, then

$H\mu_G$ $(\cap_{n=1}^{\infty}$ $A_n) = \lim_\infty \mu_G$ (A_n)

3.5 Fuzzy Regularity: For every generalized measurable fuzzy set,

μ_G $(A) = \sup\{\mu_G$ $(L) : L \subseteq A$, L generalized closed

Main Result:

4.1 Theorem: (Generalized measurability α-level sets)

Let $(X, \sum)$ be a measurable space and (γ, τ) be a topological space.

Let G: X$\rightarrow$F(γ) be a generalized fuzzy function where for every $x \in X$,

G(x) $\gamma \rightarrow [0, 1]$ is a fuzzy subset of γ. Assume that:

1. For every $\alpha \in (0,1]$, the α-level set $G_\alpha(x) = \{y \in \gamma : \mu_{F(x)}$ $(y) \geq \alpha\}$ is non- empty and closed.

2. For every open set $V \subseteq \gamma$

$\{x \in X : G_\alpha(x) \cap V \neq \phi\} \in \sum$ then G is a generalized topological fuzzy measurable function.

Proof :

Fix any open subset $V \subseteq \gamma$. By assumption

$B_\alpha(V) = \{x : G_\alpha(x) \cap V \neq \phi\}$ is measurable since

$G(x) = V_\alpha \in (0,1]$ G_α (x) the generalized inverse image of V becomes

$G^{-1}(V) = V_\alpha \in (0,1] \cap Q$ $B_\alpha(V)$

Because the rotational number in (0,1] are countable and each $B_\alpha(V)$ is measurable. $G^{-1}(V) \in \sum$

Hence G satisfies generalized fuzzy measurability. Therefore G is a generalized measurable fuzzy function.

4.2 Theorem:

Let $(X, \tau_G, \sum_G)$ be a generalized topological measurable space, and let

$\mu_G : \sum_G \geq\rightarrow [0, \infty]$

be a generalized fuzzy measure satisfying the axioms 3.1 to 3.5.

Suppose

$f_G : X\rightarrow [0,1]$ is a strongly generalized fuzzy measurable function.

Define the generalized fuzzy integral by

$M_G (f_G) = \int_0^1 \mu_G (\{x \in X : f_G(x) \geq \alpha\})d_\alpha$

then

I. $M_G (f_G)$ exists and is finite whenever $\mu_G (x) < \infty$

II. If $f_G(x) \leq g_G(x)\ \forall\ x \in X$

$M_G (f_G) \leq M_G (g_G)$

III. If $f_n \uparrow f$ point wise and every f_n is strongly generalized measurable, then

$\lim_{n\to\infty} M_G (f_G) = M_G (f)$

Proof :

By definition 2.9, f_G is strongly generalized fuzzy measurable for every $\alpha \in [0,1]$.

$E_\alpha = \{x : f_G(x) \geq \alpha\} \in \sum_G$

Hence $\mu_G(E_\alpha)$ is well defined.

by assumption/definition 2.4

Because

$0 \leq \mu_G(E_\alpha) \leq \mu_G(x)$ and $\mu_G(x) < \infty$

the mapping

$\alpha \rightarrow \mu_G(E_\alpha)$ is bounded on [0,1].

Therefore

$\int_0^1 \mu_G(E_\alpha)d_\alpha$ exists.

Hence

$M_G (f_G)$ is the existence of integral.

$f_G (x) \leq g_G(x)$ then for every level α

$\{f_G \geq \alpha\} \subseteq \{g_G \geq \alpha\}$

By monotonicity of the generalized fuzzy measure or 2.4 Definition,

$\mu_G (\{f_\alpha \geq \alpha\}) \leq \mu_G (\{g_\alpha \geq \alpha\})$ Integrating over

$\alpha \in (0,1]$ which yields

$M_G(f_G) \leq M_G (g_\alpha)$

$f_n(x) \uparrow f(x)$

then

$\{f_n \geq \alpha\} \uparrow \{f \geq \alpha\}$

By definition 2.9,

$\mu_G (\{f_n \geq \alpha\}) \rightarrow \mu_G (\{f \geq \alpha\})$

Integrating with respect to α gives

$\lim_{n\to\infty} M_G (f_G) = M_G (f)$

This complete the proof.

4.3 Theorem:

Let $(X, \sum_G, \tau_G)$ be a generalized topological measurable space and let

$\mu_G : \sum \to [0, \infty]$ be a generalized fuzzy measure.

Suppose f_G: $X \to [0,1]$ is a strongly generalized fuzzy measurable function.

I. $\tau_G = \tau$, where τ is the classical topology (τ = tau).

II. $\sum_G = \sum$, where $\sum$ is the classical σ-algebra.

III. $\mu_G = \mu$, where μ is the classical fuzzy measure and

IV. $f_G = f$, where f is a classical fuzzy measurable function.

i.e.

$(D_G) \int_x f_G \, d\mu_G = (D) \int_x f \, d\mu.$

Proof:

Since $\sum_G = \sum$ every generalized measurable set is exactly a classical measurable set. Also

$\mu_G = \mu$ (mue).

So the generalized fuzzy measure agrees with the classical measure on every measurable set.

Moreover,

$f_G = f.$

Hence, for every $\alpha \in (0,1]$,

$\{x : f_G(x) \geq \alpha\} = \{x : f(x) \geq \alpha\}.$

Therefore,

$\mu_G (\{x : f_G(x) \geq \alpha\}) = \mu (\{x : f(x) \geq \alpha\})$

Substituting into the generalized Choquet integral gives:

$(D_G) \int_x f_G \, d\mu_G = \int_0^1 \mu_G \; (\{x : f_G(x) \geq \alpha\})d\alpha$

using above equalities above,

$= \int_0^1 \mu \;\; (\{x : f(x) \geq \alpha\})d\alpha$

By the definition of classical of the Choquet fuzzy integral.

$\int_0^1 \mu \;\; (\{x : f(x) \geq \alpha\})d\alpha = (D) \int_x f \, d\mu$

Hence,

$(D_G) \int_x f_G \, d\mu_G = (D) \int_x f \; \mu$

This complete the proof.

4.4 Theorem:

Let $(P, \tau_p, \sum_p)$, $(Q, \tau_Q, \sum_Q)$, $(R, \tau_R, \sum_R)$ be topological measurable spaces. Let G_i : $P \to Q$ and j : $Q \to R$ be generalized fuzzy functions satisfying:

4.1. j is generalized fuzzy measurable.

4.2. j is generalized fuzzy measurable.

4.3. j is generalized fuzzy continuous.

4.4. Every generalized fuzzy open-subset of Q is measurable, i.e.,

$IJ\tau_Q \subseteq \sum_Q$ then

j 0 I : $P \to Q$ is a generalized fuzzy measurable function.

Proof:

We proving that the inverse image of every generalized fuzzy open set on Q is measurable in P.

Let, $G \in IJ\tau_R$

be arbitrary. Since j is generalized fuzzy continuous, the inverse image

$j^{-1}(G)$ is a generalized fuzzy open in Q.

Hence,

$j^{-1}(G) \in IJ\tau_Q$

From our assumption (4.4), every generalized fuzzy open subset of Q belongs to the measurable family σ-family

Therefore,

$j^{-1}(G) \in \Sigma_Q$

Thus the inverse image obtained through g is not only topologically generalized fuzzy open but also measurable.

Since i is generalized fuzzy measurable, the inverse image of every measurable generalized fuzzy subset of

$Q \in \Sigma_p$

Applying this property to the measurable set

$j^{-1}(G)$$, we obtain

$i^{-1}(j^{-1}(G)) \in \Sigma_p$

Now use the fundamental identity of inverse image under composition / or Lemma 2.11

$(j \text{ o } i)^{-1}(G) = j^{-1}(j^{-1}(G))$.

Since the right-hand side belongs to

Σ_p, it follows that

$(j \text{ o } i)^{-1}(G) = \in \Sigma_p$

Because the generalized fuzzy open set G use chosen arbitrarily, above property holds for every generalized fuzzy open subset of R.

Hence for every

$G \in IJ\tau_R$

we have

$(j \text{ o } i)^{-1}(G) \in \Sigma_p$

is same as the definition of generalized fuzzy measurability.

Therefore,

j o I is a generalized fuzzy measurable function.

4.5 Theorem:

Let (P, τ_p, Σ_p) and (Q, τ_Q, Σ_Q) be topological measurable spaces.

Let $I : P \rightarrow Q$ be a generalized fuzzy function.

4.5.1. i is generalized fuzzy measurable.

4.5.2. Every generalized fuzzy singleton q is measurable in Q.

4.5.3. The product measurable structure satisfies $\Sigma_p \otimes \Sigma_Q$ on $P \times Q$, the generalized fuzzy graph of i, by

$J_i = \{(p, q) \in P \times Q : q = i(p)\}$ the generalized fuzzy graph J_i is measurable in $(P \times Q, \Sigma_p \otimes \Sigma_Q)$.

Proof: - Define the mapping

$I : P \times Q \rightarrow Q \times Q$ by $I(p, q) = (i(p), q)$.

Since i is generalized fuzzy measurable and the second coordinate projection

$\phi_2(p, q) = q$ is measurable, the mapping I is measurable.

Consider the generalized fuzzy diagonal

$\Omega = \{(s,t) \in Q \times Q : s = t\}$.

By our supposition on generalized fuzzy singletons and the measurable structure of Q, assume that

$\Omega \in \Sigma_Q \otimes \Sigma_Q$ and

Observe that

$(p, q) \in I_j$ iff

$i(p) = q$, which is equivalent to

$(i(p), q) \in \Omega$ Hence,

$J_i = I^{-1}(\Omega)$ since I is measurable, the inverse image of every measurable set is measurable.

Therefore

$I^{-1}(\Omega) \in \Sigma_p \otimes \Sigma_Q$ using the identity

$J_i = I^{-1}(\Omega)$, we conclude that

$J_i \in \sum_p \otimes \sum_Q$

Hence the generalized fuzzy graph is measurable.

Corollary 4.6:

If i is generalized fuzzy continuous as well as generalized fuzzy measurable, then its graph is simultaneously

* Generalized fuzzy measurable,
* Compatible with the product measurable structure and
* Stable under generalized fuzzy inverse Image.

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