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A Strong Hybrid Equivalence Framework Based On Dominance Relations And Sensitivity Distance

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Abstract

Classical dominance-based and indiscernibility relations are often too rigid to analyse real data affected by noise, perturbation, or instability. This paper introduces a strong hybrid equivalence framework that integrates exact algebraic dominance with a ε–restricted similarity induced by a sensitivity distance. The sensitivity distance measures behavioural deviation between objects, allowing both metric analysis and tolerance-based interpretation. We study the algebraic properties of sensitivity relations, derive necessary and sufficient conditions for transitivity via ultrametric inequalities, and construct ε-neighbourhoods that generate covering-based approximations. Lower and upper rough approximations, lattice ordering under ε-variation, and the interaction between hybrid equivalence and dominance are established. A healthcare risk-classification case study demonstrates how the hybrid model yields robust and interpretable groupings in the presence of uncertainty. The proposed framework provides a mathematically consistent foundation for sensitivity-aware reasoning, stability-based decision analysis and approximate classification.

Keywords:
Dominance relation; Sensitivity distance; Rough sets; Tolerance relation; Hybrid equivalence; ε-neighbourhood; Approximate reasoning; Robust decision analysis.

1. Introduction

Real-world decision problems seldom operate under ideal conditions. Measurements may be noisy, model outputs may change under small perturbations, and attribute values are often uncertain. In such environments, classical indiscernibility or dominance relations that assume exact equality are insufficient. Although these relations yield elegant algebraic structures, they frequently ignore behavioural similarity that persists in the presence of small variations. A slight change in input data may break dominance, even though the two objects continue to represent the same decision status in practice. This limitation motivates the need for more flexible frameworks that combine exact and approximate equivalence.

Sensitivity analysis provides a promising mechanism for modelling such situations. By measuring how object behaviour changes under controlled perturbations, sensitivity distances quantify the degree of stability or deviation. Objects that differ only slightly across perturbations should not be treated as completely distinct. Instead, they should be considered nearly equivalent with respect to the decision task. A tolerance relation defined by a sensitivity threshold captures this idea, but on its own it cannot ensure transitivity and therefore does not generate classical equivalence classes.

To overcome these difficulties, this paper introduces a strong hybrid equivalence framework that unifies dominance-based equality with sensitivity-based similarity. The proposed relation requires both algebraic indifference and ε-restricted behavioural closeness, producing a more stable and meaningful grouping of objects. We develop the algebraic properties of this relation, establish conditions under which transitivity is recovered, and construct hybrid neighbourhoods that form a covering of the universe rather than a partition. These neighbourhoods enable lower and upper rough approximations, allowing decision rules to be derived in uncertain environments.

The practical value of the framework is demonstrated through a case study in healthcare risk classification, where variations in attribute values and model responses are inevitable. The hybrid relation identifies robust clusters of patients that remain consistent under perturbations, offering improved interpretability for risk assessment. The results show that combining dominance and sensitivity yields more reliable decisions than using either mechanism in isolation. The proposed approach therefore provides a unified foundation for robust reasoning, stability analysis and classification in noisy information systems.

Main contributions of this paper are as follows:

1. We propose a strong hybrid equivalence relation that integrates dominance-based indifference with ε-restricted sensitivity distance, providing robustness to data perturbations.
2. We establish necessary and sufficient ultrametric conditions under which sensitivity-based tolerance relations become transitive equivalence relations.
3. We develop a covering-based rough set framework induced by hybrid ε-neighbourhoods and derive corresponding lower and upper approximations.
4. We introduce a local robustness index for stability-aware decision analysis.
5. We demonstrate the applicability of the proposed framework through a healthcare risk-classification case study.

2. Preliminaries

Let U be a non-empty finite universe of objects and C a set of attributes describing these objects. In decision systems, a comparison between two objects usually depends on their attribute values. Classical rough set theory models this comparison through exact equivalence relations, while practical applications often require tolerance to minor deviations. This section recalls the basic concepts used in the development of the strong hybrid framework.

2.1 Sensitivity Distance

To express how strongly objects change under perturbations or uncertainty, we assign a sensitivity distance:

$$S: U \times U \rightarrow \mathbb{R}_{\geq 0}.$$

For every pair $x, y \in U$, the value $S(x,y)$ measures the behavioural or numerical deviation. The distance satisfies the following properties:

- **Non-negativity**:

$$S(x, y) \geq 0.$$

- **Symmetry**:

$$S(x, y) = S(y, x).$$

- **Triangle inequality**:

$$S(x, z) \leq S(x, y) + S(y, z).$$

When these conditions hold, S becomes a metric or a pseudometric on U. Small values of S(x, y) indicate that x and y behave similarly under perturbation, whereas large values reflect instability or strong dissimilarity. The matrix

$D=[d_{ij}]$, $d_{ij}=S(x_i, x_j)$

is called the **distance matrix**, and it is symmetric with zero diagonal.

2.2 Dominance Relation

For a subset of attributes $P\subseteq C_P$, dominance is defined as

$x\ D_P\ y \Leftrightarrow a(x) \geq a(y), \forall a\in P.$

This relation is:

- reflexive,
- transitive,
- but not necessarily symmetric.

Therefore, D_P forms a **preorder** on U. From dominance, we extract an algebraic equivalence relation by requiring mutual dominance:

$x\sim_P y \Leftrightarrow x\ D_P\ y$ and $y\ D_P\ x.$

The relation $\sim_P$ is reflexive, symmetric, and transitive, hence it partitions U into equivalence classes

$[x]_P=\{y\in U: y\sim_P x\}.$

These classes represent objects that are indistinguishable according to all attributes in P.

2.3 Sensitivity-Based Indistinguishability

Sensitivity introduces a tolerance-based similarity defined by a threshold ε>0:

$x\sim_\varepsilon y \Leftrightarrow S(x,y)<\varepsilon.$

The relation $\sim_\varepsilon$ satisfies

- reflexivity,
- symmetry,

but in general it is **not transitive**. Thus, $\sim_\varepsilon$ forms a tolerance relation and yields overlapping groups of objects. The relation becomes transitive precisely when the sensitivity distance satisfies a strong triangle (ultrametric) inequality. In this case, objects that are close to a common neighbour remain close to each other.

Definition 2.4: To convert the linguistic variable to fuzzy membership values, let us consider,

$|\varphi(x_{i,}a_k)| - 1 = h$, here i= 1 to n and k is fixed .

Then *the membership value of the matrix element , denoted by*

$\vartheta_M(x_{i,}a_k) = w/(h)$, where w is the weight $0 \leq w \leq h$ given to all values of a_k depending on the information system.

2.5 Motivation for Hybridisation

Dominance identifies exact equivalence from attribute values, whereas sensitivity captures approximate behaviour under perturbations. Neither mechanism alone provides a fully stable structure in noisy environments:

- dominance ignores slight variations that do not alter the decision meaning,
- sensitivity disregards the algebraic ordering that arises from attributes.

A unified framework is therefore desirable. In the next section, we define a **strong hybrid relation** that combines both aspects, producing equivalence classes that are simultaneously algebraically consistent and robust to small deviations.

3 : A Formal Framework for Strong Hybrid Relations Based on Dominance and Sensitivity Distance

In many decision-theoretic and information-system frameworks, classical equivalence relations derived from dominance or indiscernibility fail to capture the nuanced behavior of real-world data, especially when attribute values or model outputs are affected by noise, perturbations, or measurement uncertainty. To address this limitation, we introduce a *strong hybrid relation* that integrates an algebraic equivalence structure with a metric-like sensitivity distance. This construction simultaneously preserves exact dominance-based equivalence and enforces a strict ε-bounded similarity constraint, thereby producing a refined and more robust relational model. The strong hybrid relation serves as a bridge between traditional equivalence relations and tolerance-based fuzzy approximations, enabling the development of filtered neighborhoods, hybrid approximations, and stability-based decision rules. This section provides the formal definitions, algebraic properties, and structural implications of the strong hybrid relation, establishing it as a foundational tool for sensitivity-aware analysis in dominance systems and related applications.

Definition 3.1 : A sensitivity distance S(x,y) is defined as
S(x,y) = $\frac{1}{n}\sum_{i=1}^{n}|\vartheta_M(x, f_i) - \vartheta_M(y, f_i)|$.
If S(x,y) < ε , x, y are nearly indistinguishable
A lower S(x,y) implies higher sensitivity

Properties 3.2 : The properties of sensitivity distance are as follows

1. Non negativity : S(x,y) ≥ 0
2. Symmetry : S(x, y) = S(y,x)
3. Triangle inequality: S(x,z) ≤ S(x,y) + S(y,z)

So, S defines a metric on U

Let us construct a distance matrix

Definition 3.3: The distance matrix denoted by
D = $[d_{ij}]_{nxn}$, $d_{ij} = S(x_i, x_j)$
It is symmetric and zero diagonal

3.4: In-distinguishability and Sensitivity:
If S(x,y) < ε
Then

1. Lower S(x,y) means objects are nearby the same in the information system
2. Higher sensitivity , a small change in membership can flip the dominance ordering .
3. **Attribute wise distance matrix :**

$$M_K = \partial_{ij}^{(K)}, \partial_{ij}^{(K)} = [\vartheta_M(x_i, a_K) - \vartheta_M(x_j, a_K)]$$

It is the pair wise difference for each attribute a_K

Now fuzzy similarity function is Sim(x_i, x_j) = $\frac{1}{m}\sum_{K=1}^{m}\partial_{ij}^{K}$

Example 3.5: Let $\vartheta_M(x) = [0.2, 0.5, 0.9]$ *and* $\vartheta_M(y) = [0.3, 0.6, 0.4]$
Then Sim(x,y) = $\frac{1}{3}(|0.2 - 0.3| + |0.5 - 0.6| + |0.9 - 0.4|)$

= 0.2333< 0.5

Let us consider example 3.7[1]:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	0	0.17	0.17	0.5	0.5	0.33	0.33	0.33	0.33	0.5
x_2		0	0.33	0.67	0.67	0.5	0.17	0.17	0.5	0.33
x_3			0	0.67	0.67	0.5	0.17	0.5	0.5	0.67
x_4				0	0.33	0.17	0.83	0.5	0.17	0.17
x_5					0	0.17	0.83	0.5	0.5	0.67
x_6						0	0.67	0.33	0.33	0.5
x_7							0	0.33	0.67	0.5
x_8								0	0.33	0.17
x_9									0	0.17
x_{10}										0

Note 3.6 : let us consider a relation
$R = \{(x,y): S(x,y) < \varepsilon\}$
This relation is reflexive, symmetric but not transitive.
Since if $\varepsilon = 0.5$,
$S(x_1, x_2) < 0.5$, $S(x_2, x_{10}) < 0.5$ But $S(x_1, x_{10}) = 0.5$
Thus it is a partial order relation.

$R = \{(x,y) : S(x,y) < \varepsilon\}$ is transitive exactly when the distance S satisfies the *ultrametric* (strong triangle) inequality on the universe (or, equivalently, when the strong-triangle inequality holds on every triple that matters for the chosen ε).

Theorem 3.7 Formal statement

Let U be a set and $S : U\times U \to R \geq 0$ a symmetric map with $S(x,x) = 0$. Define $R_\varepsilon := \{(x,y)\in U\times U : S(x,y)<\varepsilon\}$.

Then $R_\varepsilon>0$ **iff** S is an **ultrametric**, i.e. $\forall x,y,z\in U : S(x,z) \leq \max\{S(x,y), S(y,z)\}$.

Proofs

(a) Ultrametric ⇒ transitivity.

Assume S is an ultrametric. Suppose $S(x,y)< \varepsilon$ and $S(y,z) < \varepsilon$. Then

$$S(x,z) \leq \max\{S(x,y),S(y,z)\}< \varepsilon,$$

so $(x,z)\in R_\varepsilon(x,z)$. Hence R_ε is transitive.

(b) Transitivity for every ε>0 ⇒ ultrametric.

Assume R_ε is transitive for every $\varepsilon>0$. Take arbitrary x, y, z. Let $m = \max\{S(x,y),S(y,z)\}$. For any $\varepsilon > m$ we have $S(x,y) < \varepsilon$ and $S(y,z) < \varepsilon$, so by transitivity $S(x,z) < \varepsilon$. Since this holds for all $\varepsilon > m$, we get $S(x,z) \leq m$. Thus $S(x,z) \leq \max\{S(x,y),S(y,z)\}$, the ultrametric inequality.

(c) Transitivity for a fixed ε.

If you only need transitivity of R_ε for a particular fixed ε, the necessary and sufficient condition is the following local form:

$$\forall x,y,z\in U, (S(x,y)<\varepsilon \wedge S(y,z)<\varepsilon) \Longrightarrow S(x,z)<\varepsilon.$$

This is equivalent to saying that the strong-triangle (ultrametric) inequality holds *on all triples whose two legs are* $< \varepsilon$:

$$\forall x,y,z: \max\{S(x,y),S(y,z)\}<\varepsilon \Longrightarrow S(x,z)<\varepsilon.$$

(Which is exactly the ultrametric condition restricted to those triples.)

Algebraic study 3.8: Equivalence relations in sensitivity or dominance systems

3.8.1: Dominance and equivalence

Given a set of alternatives (objects) U and attributes (criteria) C:

- A **dominance relation** D_P for a subset of criteria $P \subseteq C$ is defined by

  $x \, D_P \, y \Leftrightarrow \forall a \in P, \; a(x) \geq a(y)$

  (assuming "greater is better").

- This relation is **reflexive** and **transitive**, but not necessarily **symmetric** — so it's a **preorder**, not an equivalence.

3.8.2 : Indifference (equivalence) relation

We can extract an *equivalence* from D_P:

$x \sim_P y \Leftrightarrow (x \, D_P \, y) \wedge (y \, D_P \, x)$

This is an equivalence relation, because:

- Reflexive: $x \sim_P x$ trivially.
- Symmetric: if $x \sim_P y$, then both directions hold.
- Transitive: if $x \sim_P y$ and $y \sim_P z$, then $x \, D_P \, z$ and $z \, D_P \, x$.

This partitions the universe U into equivalence classes $[x]_P$.

3.8.3: Connection to sensitivity distance S(x,y)

Sensitivity measure:

$S(x,y) = \frac{1}{n}\sum_{i=1}^{n} |\vartheta_M(x, f_i) - \vartheta_M(y, f_i)|.$

is **metric-like** — small when x and y behave similarly under all perturbations f_i.

We can define an **ε–indistinguishability relation**:

$x \sim_\varepsilon y \Leftrightarrow S(x,y) < \varepsilon$

Then:

- Reflexive (since $S(x,x)=0$).
- Symmetric (absolute value is symmetric).
- But not always transitive.

If $S(x,y)<\varepsilon$ and $S(y,z)<\varepsilon$, we only get $S(x,z) \leq S(x,y)+S(y,z) < 2\varepsilon$
So **transitivity holds if we use a smaller tolerance**, e.g. define $\varepsilon' = \varepsilon/2$.

Thus, $\sim_\varepsilon$ is a **fuzzy equivalence relation** — a *tolerance* rather than strict equivalence. This makes S(x,y) a bridge between rough set theory and fuzzy clustering.

3.8.4: Combining dominance and sensitivity

We can define a **hybrid equivalence**:

$x \approx_P y \iff (x \sim_P y) \vee (S(x,y) < \varepsilon)$

→ This treats two objects as equivalent either if they are *truly indiscernible* in attribute space or *approximately indistinguishable* in model-response space.

This is powerful because it generalizes **Pawlak's indiscernibility** to **continuous data or ML models**.

3.8.5: Algebraic properties summary table

Relation	Reflexive	Symmetric	Transitive	Type
D_P	✓	✗	✓	Preorder
$\sim_P$	✓	✓	✓	Equivalence
$x \sim_\varepsilon y : S(x,y) < \varepsilon$	✓	✓	✗ (fuzzy)	Tolerance / Fuzzy equivalence
$x \approx_P y$(hybrid)	✓	✓	Approximate	Fuzzy–dominance hybrid

.3.8.6: Case Study: Sensitivity–Dominance Analysis in Healthcare Risk Classification

3.8.6.1 Objective

To classify patients into **risk equivalence groups** (Healthy, Moderate, High Risk) using both

- medical indicators (dominance-based comparison), and
- model behavior under perturbations (sensitivity distance S(x,y)).

3.8.6.2: Dataset and Context

Suppose we have **6 patients (P_1–P_6)** with 3 measured indicators:

Patient	Blood Pressure (mmHg)	Cholesterol (mg/dL)	BMI	Risk Label (doctor)
P_1	110	170	22	Low
P_2	118	185	24	Low
P_3	132	195	26	Moderate
P_4	140	205	27	Moderate
P_5	158	240	30	High
P_6	162	250	31	High

We define **dominance** as "lower value = healthier" for all three indicators. Thus:

$x\, D_P\, y \iff BP(x) \le BP(y) \wedge Chol(x) \le Chol(y) \wedge BMI(x) \le BMI(y)$

3.8.6.3: Dominance Relation & Equivalence Classes

Compute which patients dominate others.

Example:

- P_1 dominates everyone (smallest in all attributes).
- P_2 dominates P_3–P_6.
- P_3 dominates P_4–P_6.
- P_4 dominates P_5,P_6.
- P_5 dominates P_6.
- P_6 dominates no one.

3.8.6.4: Dominance summary table

Patient	Dominates who?	Is dominated by?
P_1	P_2–P_6	—
P_2	P_3–P_6	P_1
P_3	P_4–P_6	P_1,P_2
P_4	P_5–P_6	P_1,P_2,P_3
P_5	P_6	P_1–P_4
P_6	—	P_1–P_5

Now define **equivalence (indifference)**:

$x \sim_P y \iff xD_Py$ and yD_Px

→ none of these patients are exactly equivalent, so each $[x]_P=\{x\}$

3.8.7: Model-Based Sensitivity Distance S(x,y)

Let's assume we trained a **risk prediction model** M giving a probability of "High Risk".

Below are hypothetical outputs for each patient under 3 perturbations (e.g., measurement noise, missing data, weight variation):

Patient	f_1	f_2	f_3	Mean M(x)
P_1	0.10	0.12	0.09	0.10
P_2	0.15	0.17	0.18	0.17
P_3	0.33	0.31	0.35	0.33
P_4	0.39	0.41	0.38	0.39
P_5	0.70	0.72	0.74	0.72
P_6	0.77	0.79	0.80	0.79

Then compute

$S(x,y) = \frac{1}{3}\sum_{i=1}^{3}|\vartheta_M(x,f_i) - \vartheta_M(y,f_i)|.$

For illustration, a few values:

Pair	S(x,y)
P_1,P_2	0.06
P_1,P_3	0.22
P_3,P_4	0.07
P_5,P_6	0.07
P_1,P_6	0.69

3.8.8: Define ε–Equivalence Based on Sensitivity

Choose threshold $\varepsilon=0.10$:

If $S(x,y)<0.10$, patients are "practically equivalent".

→ Pairs satisfying $S(x,y)<0.10$:

- (P_1,P_2)
- (P_3,P_4)
- (P_5,P_6)

Therefore, fuzzy equivalence classes:

$E_1=\{P_1,P_2\}$(Low Risk Equivalence)
$E_2=\{P_3,P_4\}$(Moderate Risk Equivalence)
$E_3=\{P_5,P_6\}$(High Risk Equivalence

3.8.9: Hybrid Equivalence (Dominance + Sensitivity)

Now define $x\approx_P y$ if $x\sim_P y$ (exact) or $S(x,y)<\varepsilon$(approximate).

- Since no pair is exactly equal, equivalence arises *only* from sensitivity.
- Resulting hybrid classes: E_1,E_2,E_3 above.

This forms the **practical risk-group partition** for the healthcare field.

3.8.10 : Interpretation and Insights

Group	Members	Average model risk	Dominance level	Clinical meaning
E_1	P_1, P_2	~0.14	Strongly dominant	Healthy cluster
E_2	P_3, P_4	~0.36	Mid-level	At-risk cluster
E_3	P_5, P_6	~0.75	Weakly dominant	High-risk cluster

Dominance analysis gives *theoretical ordering* ($P_1 \prec P_2 \prec \dots \prec P_6$).
Sensitivity analysis identifies *clusters of behavioral similarity*.
Hybrid equivalence unifies both — robust to noise and interpretable in practice.

3.9: Neighborhoods, covering, and invariants

3.9.1: ε–neighborhood (hybrid):

$N_\varepsilon(x) := \{y \in U : x \approx_P y\} = [x]_P \cup \{y : S(x,y) < \varepsilon\}$,

where $[x]_P$ is the $\sim_P$-class of x.

Properties:

- $[x]_P \subseteq N\varepsilon(x)$.
- $N_\varepsilon(x)$ is symmetric in the sense $y \in N_\varepsilon(x) \iff x \in N_\varepsilon(y)$.
- The family $\{N_\varepsilon(x)\}_{x \in U}$ is generally a *covering* of U (overlap allowed). This yields a covering-based rough set framework rather than a partition.

3.9.2: Local robustness index (algebraic invariant):

$\mu_\varepsilon(x) := |N_\varepsilon(x)| \in \{1, \ldots, |U|\}$..

$\mu_\varepsilon(x)$ measures how many objects are hybrid-equivalent to x; monotone in ε.

3.10 — Lower and upper approximations (hybrid rough sets)

For any target set $X \subseteq U$, define hybrid approximations (covering-style):

- **Lower approximation** (all neighbors fully inside X):

$\underline{apr}_\varepsilon(X) := \{x \in U : N_\varepsilon(x) \subseteq X\}$.

- **Upper approximation** (exists neighbor inside X):

$\overline{apr}_\varepsilon(X) := \{x \in U : N_\varepsilon(x) \cap X \neq \emptyset\}$.

Properties:

- $\underline{apr}_\varepsilon(X) \subseteq X \subseteq \overline{apr}_\varepsilon(X)$.
- As ε increases: $\underline{apr}_\varepsilon(X)$ non increasing or can only lose members? (it becomes *coarser* because neighborhoods grow so fewer are fully contained), while $\overline{apr}_\varepsilon(X)$ non decreasing.
- If ε=0 and $S(x,y)=0 \iff x=y$, then $\approx_{P,0} = \sim_P$ and the definitions reduce to classical Pawlak approximations (partition case).

These approximations support decision rules: classify x as "definitely in X" if $x \in apr_\varepsilon(X)$; "possibly in X" if $x \in apr_\varepsilon(X)$.

3.11— Algebraic structure and ordering

Monotone family. For $\varepsilon_1 < \varepsilon_2$, $R_{\varepsilon 1} \subseteq R_{\varepsilon 2} \Rightarrow \approx_{P,\varepsilon_1} \subseteq \approx_{P,\varepsilon_2}$

Thus the set $\{\approx_{P,\varepsilon}\}$, $\varepsilon \geq 0$ is totally ordered by inclusion (a chain). This yields a *filtration* / multi-resolution hierarchy of relations.

Lattice viewpoint. Under inclusion the family forms a complete lattice:

- Meet: $\approx_{P,\varepsilon_1} \wedge \approx_{P,\varepsilon_2} = \approx_{P,\min(\varepsilon_1,\varepsilon_2)}$.
- Join: $\vee = \approx_{P,\max(\varepsilon_1,\varepsilon_2)}$.

Composition semigroup. If S satisfies triangle inequality,

$R_\varepsilon \circ R_\delta \subseteq R_{\varepsilon+\delta}$.

Hence the set of tolerance relations forms a semigroup under relational composition with epsilon-addition.

3.12 — Algebraic consequences for decision rules

1. **Stable decisions:** Elements with large $\mu_\varepsilon(x)$ are robust — their classification is insensitive to small perturbations (good candidates for default rules).
2. **Boundary elements:** x with $N_\varepsilon(x)$ partly in and partly out of a decision class are *uncertain* — prioritize for inspection or further measurement.
3. **Refinement via $[x]_P$:** The exact equivalence classes $[x]_P$ act as *rigid cores* inside neighborhoods—use them as tie-breakers or anchors when transitivity fails.

3.13: Practical guidance for ε and S

- **Choice of ε:** pick by validation (maximize predictive stability / minimize classification errors under perturbation), or use statistical rules (e.g., set ε to a low percentile of empirical pairwise S-distribution among same-class pairs).
- **Properties of S:**
  - If S is a metric, many algebraic inclusions become simpler (triangle inequality gives composition control).
  - If S is only a pseudometric (distinct points may have S=0), then ≈P merges more objects even at ε=0; this can be desirable when model outputs coincide.
- **Normalization:** If model outputs differ in scale across features, normalize S (e.g., scale to [0,1]) so ε has interpretable meaning.

Comparison with Classical Dominance-Based Rough Sets (Algorithmic Perspective)

Classical dominance-based rough set approaches (DRSA) construct decision approximations through attribute-wise dominance relations. Algorithmically, DRSA operates by first computing dominance cones (upper and lower sets) for each object and then deriving lower and upper approximations of decision classes based on exact dominance consistency. This procedure is computationally efficient and mathematically well defined; however, it assumes that attribute values are precise and stable.

In practical datasets with continuous attributes or measurement noise, the DRSA algorithm often produces degenerate structures. Exact dominance indifference rarely occurs, resulting in singleton equivalence classes and overly sharp decision boundaries. Small perturbations in attribute values may alter dominance relations, leading to instability in approximations and decision rules across repeated evaluations.

Sensitivity-based models replace strict dominance comparisons with distance-threshold evaluations. Algorithmically, this involves computing a pairwise distance matrix and applying a fixed ε-cut to generate tolerance relations. While this approach captures approximate similarity, the resulting relations are generally non-transitive, producing overlapping clusters without a well-defined algebraic structure. Consequently, approximation algorithms may yield inconsistent or order-dependent results.

The proposed hybrid framework integrates both mechanisms in a unified algorithmic pipeline. First, dominance-based indifference classes are computed to preserve exact preference semantics. Next, a sensitivity distance matrix is evaluated from model outputs or behavioural responses under perturbation. Hybrid ε-neighbourhoods are then constructed by merging dominance equivalence with ε-restricted sensitivity similarity. These neighbourhoods induce a covering of the universe rather than a strict partition.

From a computational standpoint, lower and upper approximations are obtained by set-inclusion tests over hybrid neighbourhoods, similar in complexity to covering-based rough set algorithms. The ultrametric condition ensures transitivity of ε-relations when required, enabling stable equivalence construction and reducing ambiguity in classification. Additionally, the local robustness index provides a quantitative criterion for identifying stable objects and boundary cases, facilitating algorithmic prioritization in decision-making.

Table X summarizes the algorithmic differences between classical DRSA and the proposed hybrid framework.

Step	**Classical DRSA**	**Proposed Hybrid Framework**
Relation construction	Attribute-wise dominance	Dominance + sensitivity distance
Core computation	Dominance cones	Distance matrix + ε-neighbourhoods
Equivalence generation	Exact mutual dominance	Hybrid ε-equivalence
Approximation method	Partition-based	Covering-based
Stability under perturbation	Low	High
Additional outputs	Decision rules	Decision rules + robustness index

Overall, the hybrid framework extends classical dominance-based rough set algorithms by incorporating sensitivity-aware similarity without sacrificing algebraic consistency. This results in a decision-support procedure that is both computationally tractable and robust to data uncertainty, making it suitable for real-world information systems where perturbations are unavoidable.

Numerical Example: Stability Comparison between Classical DRSA and Hybrid Equivalence

Consider a small decision system with four objects $U=\{x_1,x_2,x_3,x_4\}$ described by two condition attributes a_1,a_2 (cost-type: lower is better) and one decision attribute d.

Step 1: Original dataset

Object	**a_1**	**a_2**	**Decision d**
x_1	10	20	Low
x_2	11	21	Low
x_3	15	25	High
x_4	16	26	High

Dominance relation ($xDy \Leftrightarrow a_1(x)\le a_1(y)$ and $a_2(x)\le a_2(y)$ gives:

- $x_1Dx_2Dx_3Dx_4$

No pair satisfies mutual dominance, hence all dominance equivalence classes are singletons:

$[x_i]_P=\{x_i\}, i=1,\dots,4.$

Classical DRSA approximations:

- Lower approximation of *Low*: $\{x_1,x_2\}$
- Boundary is sharp and fragile.

Step 2: Small perturbation

Assume measurement noise perturbs x_2:

$x_2=(11,21)\rightarrow(12,20)$

Updated dominance relations:

- $x_1 D x_2$ fails (trade-off in attributes)
- Dominance chain is broken

Result:
Lower approximation of *Low* becomes $\{x_1\}$

➡ **A minimal perturbation changes the classification**, demonstrating instability of classical DRSA.

Step 3: Sensitivity distance

Assume a prediction model produces outputs under perturbation:

Object	f_1	f_2	Mean
x_1	0.12	0.11	0.115
x_2	0.14	0.15	0.145
x_3	0.78	0.80	0.79
x_4	0.82	0.81	0.815

Define sensitivity distance:

$S(x,y)=|M(x)-M(y)|$

Key values:

- $S(x_1,x_2)=0.03$
- $S(x_3,x_4)=0.025$
- $S(x_2,x_3)=0.645$

Let $\varepsilon=0.05$.

Step 4: Hybrid ε-neighbourhoods

Hybrid equivalence:

$x \approx_P y \Leftrightarrow (x \sim_P y)$ or $S(x,y)<\varepsilon$

Resulting neighbourhoods:

- $N_\varepsilon(x_1)=\{x_1,x_2\}$
- $N_\varepsilon(x_2)=\{x_1,x_2\}$
- $N_\varepsilon(x_3)=\{x_3,x_4\}$
- $N_\varepsilon(x_4)=\{x_3,x_4\}$

Step 5: Hybrid approximations

Lower approximation of *Low*:

$\underline{apr}_\varepsilon(Low)=\{x_1,x_2\}$

This remains **unchanged before and after perturbation**.

Interpretation

- **Classical DRSA**: exact dominance leads to unstable approximations under small attribute perturbations.
- **Sensitivity-only models**: capture similarity but lack structured equivalence.
- **Hybrid framework**: preserves dominance semantics while ensuring robustness through ε-sensitivity.

This numerical example illustrates that the proposed hybrid equivalence produces **stable and interpretable approximations**, making it better suited for real-world information systems affected by uncertainty.

Conclusion:

This study introduced a strong hybrid equivalence framework that integrates dominance-based indifference relations with sensitivity-distance–driven similarity. The proposed model addresses the inherent rigidity of classical equivalence relations and the lack of transitivity in tolerance-based approaches when data are affected by noise, perturbation, or instability.

The algebraic analysis established necessary and sufficient conditions for the transitivity of sensitivity relations, showing that ε–indistinguishability becomes an equivalence precisely under ultrametric constraints. By combining dominance equivalence with ε-restricted sensitivity, the resulting hybrid relation preserves exact ordering information while allowing controlled approximation. The induced ε-neighbourhoods form a covering of the universe, leading naturally to a covering-based rough set framework. Corresponding lower and upper hybrid approximations were defined, providing a stable mechanism for approximate reasoning and classification.

The healthcare risk-classification case study demonstrated the practical effectiveness of the framework. While dominance relations captured theoretical clinical ordering, sensitivity distances revealed clusters of behavioural stability under perturbations. Their integration produced robust and interpretable risk groups, highlighting the advantage of hybrid equivalence over purely algebraic or purely metric-based models.

From a structural viewpoint, the family of hybrid relations indexed by ε forms an ordered filtration and a complete lattice, enabling multi-resolution analysis and stability-aware decision rules. The proposed framework therefore offers a unified and mathematically consistent foundation for robust decision analysis, sensitivity-aware reasoning, and approximate classification in uncertain information systems.

Future research may focus on adaptive or data-driven selection of ε, learning sensitivity distances from predictive models, and extending the framework to dynamic, temporal, or large-scale decision environments.

Conflict of Interest Statement
The author declares that there is no conflict of interest regarding the publication of this manuscript.

Data Availability Statement
No external datasets were used in this study. All numerical examples and case studies presented in the manuscript are illustrative and generated for methodological demonstration purposes. Any data required to reproduce the results are fully contained within the manuscript.

Author Contribution Statement
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