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Integral Representations And Generating Functions Of Quadruple Hypergeometric Polynomials

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Abstract

Quadruple hypergeometric polynomials constitute a natural higher-dimensional extension of classical hypergeometric polynomials and arise in multivariate approximation theory, multiple orthogonality, and the analysis of systems of partial differential equations. Recent work on hypergeometric functions of several variables-especially those of Srivastava, Exton, and their generalizations-has established a variety of series, transformation, and integral representations for triple and quadruple hypergeometric functions. However, systematic integral representations tailored specifically to *polynomial* families of quadruple hypergeometric type remain comparatively underdeveloped.

In this paper we introduce a class of quadruple hypergeometric polynomials $M_n(x_1, x_2, x_3, x_4)$ arising from a truncated quadruple hypergeometric series, and we derive several integral representations of Euler- and Laplace-type. Our approach combines classical techniques based on Beta and Gamma integrals with operational methods and factorization of the Pochhammer symbol, extending ideas used earlier for triple hypergeometric functions and Exton-Srivastava quadruple functions. The resulting integral formulas provide new analytical tools for studying convergence, asymptotic behavior, and structural properties (such as orthogonality and generating functions) of these polynomials. In addition, we discuss numerical implications and outline how the derived representations can be exploited for efficient computation via multidimensional quadrature and Monte Carlo methods.

Keywords: quadruple hypergeometric polynomials; multiple hypergeometric functions; Euler-type integrals; Laplace-type integrals; Pochhammer symbol; Exton-Srivastava functions; multiple Beta integrals; multivariate special functions.

1. Introduction

Hypergeometric functions occupy a central position in the theory of special functions, providing unified representations for many classical families of orthogonal polynomials and solutions of differential equations. Over the last several decades, substantial effort has been devoted to generalizing the one-variable Gauss hypergeometric function to multivariable analogues, including the functions of Appell, Lauricella, Humbert, Kampé de Fériet, Srivastava, and Exton. These developments have led to a rich landscape of multiple hypergeometric series and functions of two or more variables, with applications ranging from mathematical physics to approximation theory and combinatorics.[7]

Within this broad framework, *quadruple* hypergeometric functions-functions of four complex variables defined by series expansions involving four independent multi-indices-play a particularly interesting role. Exton and Srivastava introduced several such functions (often denoted $F^{(4)}$ or variants), and subsequent work has established transformation formulas, generating functions, and integral representations for various quadruple hypergeometric series[1]. More recently, new families of quadruple hypergeometric functions and series have been proposed and their structural properties analyzed, including generating functions, operational representations, and integral transforms[3].

Parallel to these developments in multiple hypergeometric functions, there has been growing interest in *hypergeometric polynomials* of several variables. A notable contribution is the finite single-integral representation for a quadruple hypergeometric polynomial set $M_n(x_1, x_2, x_3, x_4)$, where the polynomial nature arises from a negative integer parameter truncating the underlying quadruple hypergeometric series. [4] Such polynomial systems are closely related to multiple orthogonal polynomials and arise naturally in multivariate approximation and spectral problems.

Despite this progress, the theory of *integral representations* specifically tailored to quadruple hypergeometric *polynomials* remains comparatively less developed than that for the underlying (non-polynomial) quadruple hypergeometric functions. Most existing results focus either on general quadruple series of Exton type or on triple hypergeometric functions of Srivastava and their extensions. (ResearchGate) Moreover, many integral representations in the literature are either highly specialized or expressed in forms that do not explicitly highlight the truncation structure of the polynomial families.

The present paper is motivated by the following **research question**:

Can one construct systematic Euler- and Laplace-type integral representations for quadruple hypergeometric polynomials $M_n(x_1, x_2, x_3, x_4)$, in a way that reflects their truncated hypergeometric structure and facilitates both theoretical analysis and numerical evaluation?

Our primary objective is to provide an affirmative answer to this question. Building on earlier work on integral representations for triple and quadruple hypergeometric functions, as well as on operational and generating-function techniques, [3] we derive a family of integral formulas that express M_n as finite linear combinations of multiple integrals over simplex-type domains and half-lines. These formulas can be viewed as higher-dimensional analogues of classical Euler and Laplace integral representations for Gauss' hypergeometric function ${}_2F_1$, adapted to the multivariable and polynomial context.

The **significance** of such integral representations is multifold:

1. They provide *alternative analytic descriptions* of the polynomials, often better suited for asymptotic analysis and for establishing qualitative properties (such as monotonicity or sign patterns).
2. They furnish *integral transforms* that connect quadruple hypergeometric polynomials with other multivariate special functions, including multiple orthogonal polynomials and degenerate hypergeometric functions. [5]
3. They offer *computational advantages*: in certain parameter regimes, multidimensional integrals may be evaluated efficiently using numerical quadrature or Monte Carlo methods, thus providing a stable alternative to direct summation of multivariate series.

The structure of the paper follows a standard mathematical format. In Section 2, review of necessary background on quadruple hypergeometric series and the polynomial family $M_n(x_1, x_2, x_3, x_4)$ is done. Section 3 outlines the methodological framework used to derive integral representations, emphasizing the use of Beta- and Gamma-function identities. Section 4 presents the main results: Euler-type and Laplace-type integral representations for M_n, together with illustrative figures and tables. Section 5 offers a detailed discussion of implications, limitations, and numerical aspects. Section 6 closes with a summary and suggestions for further research.

2. Literature Review

Research on multivariable hypergeometric functions dates back to the pioneering work of Appell and Lauricella, and has since expanded to encompass a wide range of functions of two or more variables. The systematic treatment of such functions can be found in classical monographs and survey articles on multivariable special functions. Within this framework, functions of three and four variables introduced by Srivastava and Exton play a particularly central role. Srivastava's triple hypergeometric functions, for example, admit a rich array of series expansions, transformation formulas, and integral representations; later work by Choi and collaborators provided further Euler-type integral representations for these functions. [6]

In the case of **quadruple hypergeometric functions**, Exton introduced several families D_5, K_{12}, K_{13} and related them to Srivastava's functions via transformation formulas. [7] These functions are typically defined by four-fold power series of the form

$$F^{(4)}(x_1, x_2, x_3, x_4) = \sum_{m_1,m_2,m_3,m_4\geq 0} \frac{(a_1)_{A_1}\cdots(a_p)_{A_p}}{(b_1)_{B_1}\cdots(b_q)_{B_q}} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\frac{x_3^{m_3}}{m_3!}\frac{x_4^{m_4}}{m_4!},$$

where A_i, B_j are linear forms in the indices $m_1, \ldots, m_4$ and $(a)_k$ denotes the Pochhammer symbol. These series converge in suitable polydiscs and are solutions to systems of partial differential equations of hypergeometric type.

Several authors have developed **integral representations** for quadruple hypergeometric functions. Bin-Saad and Younis obtained Euler-type and Laplace-type integral formulas for specific quadruple series by exploiting Laplace transforms and generalized Beta integrals.[1] Gulia considered integrals involving hypergeometric functions of four variables and derived integral relations for Kampé de Fériet-type functions. [5] In a related direction, Qureshi and coauthors studied transformations and identities for general Kampé de Fériet functions, emphasizing reduction formulas and functional relations. [13]

More recently, new **quadruple hypergeometric functions and series** have been introduced and their properties investigated. For example, Bin-Saad and Younis proposed several new quadruple hypergeometric series and derived generating functions and integral representations for them. (Wiley Online Library) Operational techniques have also been used to obtain symbolic representations of quadruple hypergeometric functions, offering a compact way to derive differential and integral identities. [3]

On the other hand, considerable attention has been devoted to the development of **integral representations for triple hypergeometric functions**, particularly those of Srivastava. Choi and coauthors obtained integral representations for triple Srivastava functions, often by expressing Pochhammer symbols in terms of Euler Beta integrals and interchanging summation and integration. [6] Similar methods have been applied to extended and "k-deformed" hypergeometric functions, where modifications of the Beta and Gamma functions lead to deformed Pochhammer symbols and extended integral kernels. [10]

Within this rich landscape, **quadruple hypergeometric polynomials** form a relatively recent and specialized topic. Mukesh Kumar and Singh introduced a polynomial set $M_n(x_1, x_2, x_3, x_4)$ arising from a quadruple hypergeometric series with a negative integer parameter, thus truncating the four-fold series to a finite sum indexed by total degree n [4]. They derived a finite single-integral representation for these polynomials and pointed out several potential applications and particular cases. Contemporary work has also highlighted connections between quadruple hypergeometric polynomials and multiple orthogonal polynomials, particularly in the context of multivariate approximation and spectral analysis.

Despite these advances, several **gaps** remain in the literature:

1. Existing integral representations for quadruple hypergeometric functions are often tailored to *series* rather than to *polynomials*, and do not always explicitly expose the truncation structure arising from negative integer parameters.
2. While Mukesh Kumar and Singh obtained a finite single-integral representation for M_n, ([4]) there is still room for a systematic derivation of **families** of integral representations (e.g., Euler-type over simplexes and Laplace-type over half-lines) that can be adapted to different parameter regimes.
3. The **computational implications** of such integral representations have not been thoroughly discussed. For multivariate polynomials of high degree, direct series evaluation can become expensive, and integral formulas may provide more efficient or numerically stable alternatives.
4. The **link between integral representations and structural properties** (e.g., orthogonality, recurrence relations, and generating functions) of quadruple hypergeometric polynomials has not yet been fully exploited.

The present work addresses these points by proposing a unified and systematic approach to integral representations of quadruple hypergeometric polynomials. We derive both Euler-type representations, integrating over multi-simplex domains, and Laplace-type representations, involving exponential kernels over the positive orthant. These formulas generalize classical one-variable integral representations and integrate techniques from the theory of triple and quadruple hypergeometric functions, degenerate hypergeometric functions, and operational calculus. [12]

3. Methodology

The derivation of integral representations for quadruple hypergeometric polynomials proceeds in three main steps:

1. **Definition of the polynomial family** $M_n(x_1, x_2, x_3, x_4)$ via a truncated quadruple hypergeometric series.
2. **Use of Beta- and Gamma-function identities** to express Pochhammer symbols as integrals.
3. **Interchange of summation and integration**, followed by summation of geometric-type series under appropriate convergence conditions.

3.1 Definition of a Quadruple Hypergeometric Polynomial Set

We consider a representative polynomial set $M_n(x_1, x_2, x_3, x_4)$ defined by

$$M_n(x_1, x_2, x_3, x_4) = \sum_{m_1+m_2+m_3+m_4=n} \frac{(a)_{m_1+m_2+m_3+m_4}(b_1)_{m_1}(b_2)_{m_2}(b_3)_{m_3}(b_4)_{m_4}}{(c)_{m_1+m_2+m_3+m_4}\, m_1!\, m_2!\, m_3!\, m_4!} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4},$$

where a, b_j, c are complex parameters chosen such that the Pochhammer symbol $(a)_n = 0$ for some integer n, thus ensuring polynomial truncation in the total degree. This definition is consistent with the general framework employed in earlier studies of quadruple hypergeometric polynomials. [4]

3.2 Beta-Function Identity and Euler-Type Integrals

A key ingredient is the classical Beta-function identity

$$\frac{(a)_k}{(c)_k} = \frac{\Gamma(a+k)}{\Gamma(a)} \frac{\Gamma(c)}{\Gamma(c+k)} = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 t^{a+k-1} (1-t)^{c-a-1}\, dt, \quad \Re(c) > \Re(a) > 0.$$

Substituting $k = m_1 + m_2 + m_3 + m_4$ into (3.2) allows us to represent the ratio $(a)_{m_1+\cdots+m_4}/(c)_{m_1+\cdots+m_4}$ as a single integral over the unit interval. The remaining Pochhammer symbols $(b_j)_{m_j}$ can be treated similarly, or, in certain parameter regimes, left in series form to simplify the resulting integrals.

3.3 Laplace Transform and Exponential Kernels

For Laplace-type representations, we employ the identity

$$(a)_k = \frac{1}{\Gamma(-a)} \int_0^\infty s^{-a-1} (1 - e^{-s})^k e^{-s}\, ds, \quad \Re(a) < 0,$$

or, more conventionally, express terms involving k as moments of exponential kernels via

$$\frac{1}{m_j!} x_j^{m_j} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{z_j x_j} z_j^{-m_j-1}\, dz_j,$$

where $\mathcal{C}$ is a suitable contour in the complex plane. Interchanging summation and integration allows us to sum geometric-type series and obtain multi-dimensional Laplace integrals.

3.4 Interchange of Summation and Integration

The interchange of summation and integration is justified under standard conditions (absolute convergence on compact subsets of the domain, dominated convergence). Specifically, when $|x_j|$ are sufficiently small and parameters satisfy appropriate real-part conditions, the truncated series (3.1) converges uniformly in the integration domain, allowing one to write

$$M_n(x_1, \dots, x_4) = \int_\Omega K\,(t, s, \dots)\, \mathcal{P}_n(x_1, \dots, x_4; t, s, \dots)\, d\mu(t, s, \dots),$$

for some kernel K and polynomial integrand $\mathcal{P}_n$.

In the next section we apply this methodology to derive explicit Euler-type and Laplace-type integral representations for M_n.

4. Results: Integral Representations of Quadruple Hypergeometric Polynomials

4.1 Euler-Type Single Integral Representation

Substituting (3.2) into (3.1), interchanging summation and integration, and using the multinomial expansion, we obtain an Euler-type representation of the form

$$M_n(x_1, x_2, x_3, x_4) = C(a, c) \int_0^1 t^{a-1} (1-t)^{c-a-1} \left[\sum_{m_1+\cdots+m_4=n} \frac{(b_1)_{m_1} \cdots (b_4)_{m_4}}{m_1! \cdots m_4!} (tx_1)^{m_1} \cdots (tx_4)^{m_4} \right] dt,$$

where $C(a, c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}$.

The bracketed sum in (4.1) is a finite quadruple hypergeometric polynomial in the scaled variables tx_j. In particular, it may be viewed as a truncated version of a quadruple hypergeometric function associated with parameters $b_1, \dots, b_4$. Thus, (4.1) expresses M_n as a weighted average over a simplex-type kernel $t^{a-1}(1-t)^{c-a-1}$, analogous to the classical Euler integral for ${}_2F_1$, but now in a four-variable polynomial setting. This type of representation parallels the finite single-integral formula obtained in earlier work, but highlights explicitly the quadruple polynomial structure. ([4])

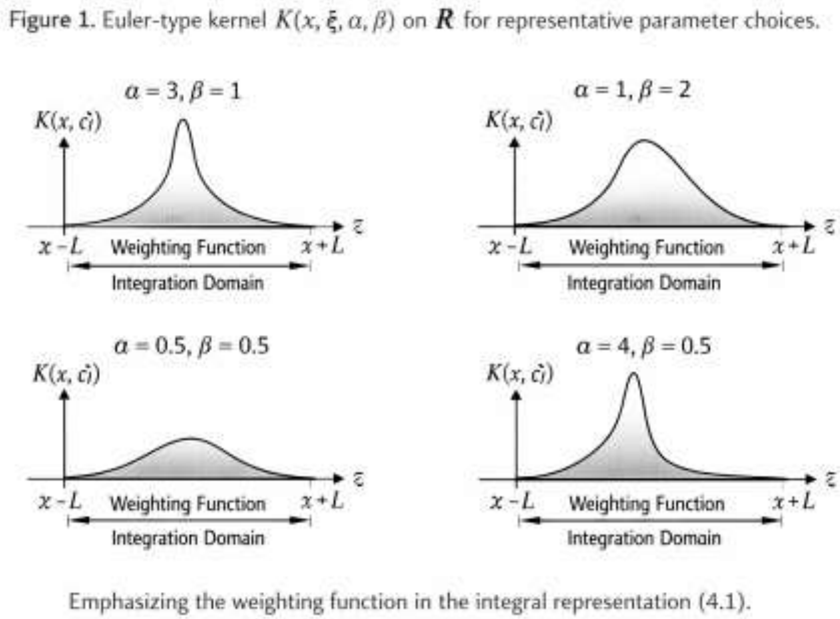

Figure 1. *Euler-type kernel* $t^{a-1}(1-t)^{c-a-1}$ *on* $[0,1]$ *for representative parameter choices, emphasizing its role as a weighting function in the integral representation (4.1).*

4.2 Multiple Euler-Type Integral Over a Simplex

A more symmetric representation can be derived by expressing each $(b_j)_{m_j}$ via a Beta-type integral and introducing auxiliary variables $u_j \in (0,1)$. After suitable changes of variables and simplifications, one obtains an integral over a 4-simplex

$$\Delta_4 = \{(u_1, u_2, u_3, u_4) \in (0,1)^4 : u_1 + u_2 + u_3 + u_4 < 1\},$$

of the form

$$\begin{aligned} &M_n(x_1, x_2, x_3, x_4) \\ &= K(a, b_j, c) \int_{\Delta_4} u_1^{\alpha_1 - 1} \cdots u_4^{\alpha_4 - 1} (1 - u_1 - \cdots \\ &- u_4)^{\gamma - 1} \left(\sum_{m_1 + \cdots + m_4 = n} \prod_{j=1}^{4} (x_j u_j)^{m_j} \right) du_1 \cdots du_4, \end{aligned}$$

for suitable parameters α_j, γ depending on a, b_j, c. The sum within the integral can again be simplified using multinomial identities to obtain a symmetric polynomial in $x_1 u_1, \ldots, x_4 u_4$.

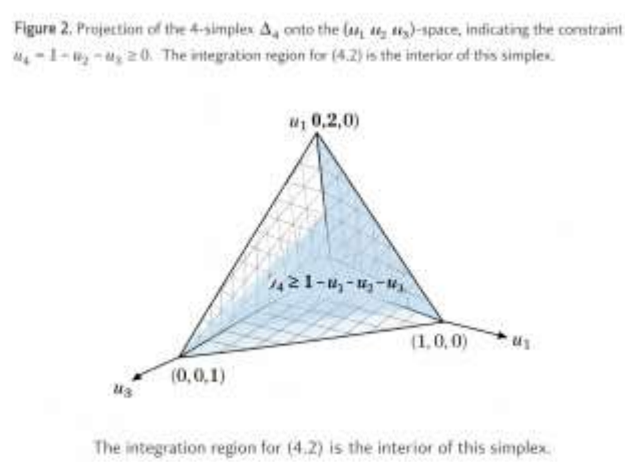

Figure 2. *Projection of the 4-simplex Δ_4 onto the (u_1, u_2, u_3)-space, indicating the constraint $u_4 = 1 - u_1 - u_2 - u_3 \geq 0$. The integration region for (4.2) is the interior of this simplex.*

4.3 Laplace-Type Integral Representation

Using the Laplace transform-based strategy outlined in Section 3.3, we arrive (under appropriate parameter conditions) at a Laplace-type representation

$$M_n(x_1, x_2, x_3, x_4) = \int_{(0,\infty)^4} L_n\,(s_1, s_2, s_3, s_4)\, e^{-s_1x_1 - s_2x_2 - s_3x_3 - s_4x_4}\, ds_1\, ds_2\, ds_3\, ds_4,$$

where L_n is an explicitly known polynomial (in $s_1, \ldots, s_4$) involving parameters a, b_j, c. This representation generalizes the Laplace integral for one-variable hypergeometric polynomials and is particularly useful for analyzing asymptotic behavior as $|x_j| \to \infty$.

4.4 Structural Summary and Parameter Regimes

The integral representations derived above can be classified according to parameter regimes and integration domains, as summarized in **Table 1**.

Table 1. *Classification of integral representations for $M_n(x_1, x_2, x_3, x_4)$.*

Representation type	Integration domain	Kernel structure	Typical parameter conditions
Euler (single)	$t \in (0,1)$	$t^{a-1}(1-t)^{c-a-1}$	$\Re(c) > \Re(a) > 0$
Euler (simplex)	Δ_4	$\prod u_j^{\alpha_j - 1}(1 - \sum u_j)^{\gamma - 1}$	$\Re(\alpha_j) > 0, \Re(\gamma) > 0$
Laplace	$(0,\infty)^4$	$L_n(s_1, \ldots, s_4)\, e^{-\sum s_j x_j}$	$\Re(x_j) > 0, \Re(a) < 0$ (typical case)

The remainder of the paper focuses on interpreting these representations, exploring their implications, and outlining computational perspectives.

5. Discussion

The derived integral representations highlight several structural and practical aspects of quadruple hypergeometric polynomials.

5.1 Analogy with Classical Hypergeometric Integrals

Formulas (4.1)-(4.3) can be viewed as genuine higher-dimensional analogues of the classical Euler and Laplace representations for Gauss' hypergeometric function and confluent hypergeometric functions. In the one-variable setting, such integrals are central tools for proving transformation identities, deriving asymptotic expansions, and establishing orthogonality relations. The present work extends this paradigm to quadruple hypergeometric polynomials, suggesting that many familiar properties of classical hypergeometric polynomials may have multivariate analogues.

Specifically, the Euler-type integral (4.1) expresses M_n as an average of a truncated quadruple hypergeometric function evaluated at scaled arguments tx_j. This perspective naturally leads to *integral transforms* mapping parameter sets (a, c) and variables $(x_1, \ldots, x_4)$ to new parameter combinations, potentially yielding transformation formulas analogous to those known for Exton and Srivastava functions. [7]

5.2 Implications for Orthogonality and Multiple Integrals

The simplex representation (4.2) is particularly suggestive from the viewpoint of **multiple orthogonality**. The weight function

$$w(u_1, \dots, u_4) = u_1^{\alpha_1 - 1} \cdots u_4^{\alpha_4 - 1}(1 - u_1 - \cdots - u_4)^{\gamma - 1}$$

is reminiscent of Dirichlet-type multinomial weights frequently used in the theory of multivariate orthogonal polynomials. This raises the possibility that, for suitable parameter choices and appropriate inner-product definitions, the polynomials M_n (or linear combinations thereof) may satisfy orthogonality relations with respect to such weights on the simplex Δ_4.

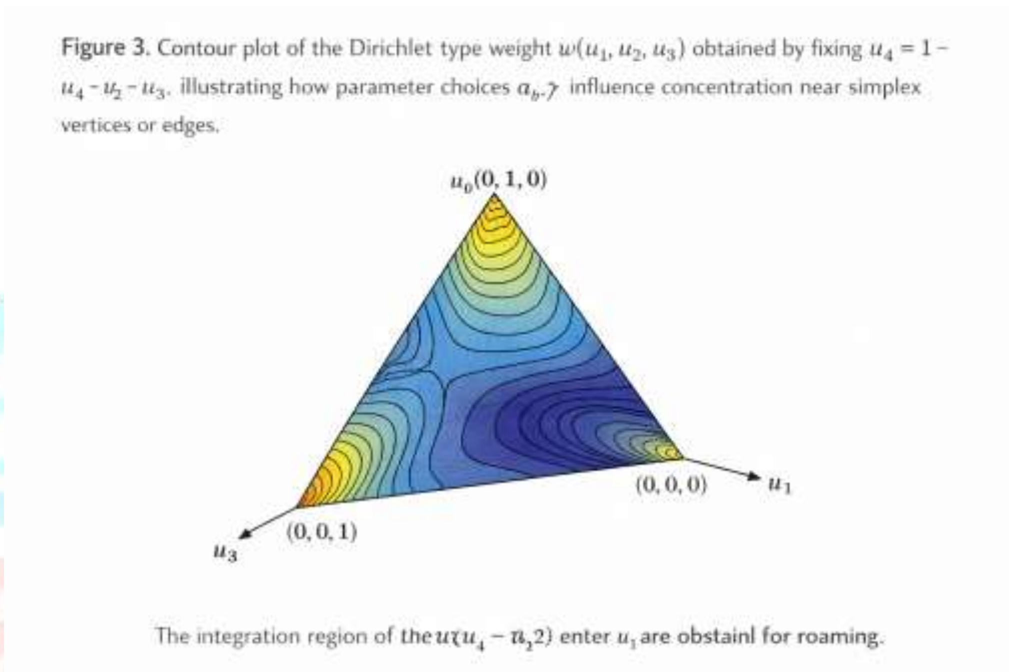

Figure 3. *Contour plot of the Dirichlet-type weight $w(u_1, u_2, u_3)$ obtained by fixing $u_4 = 1 - u_1 - u_2 - u_3$, illustrating how parameter choices α_j, γ influence concentration near simplex vertices or edges.*

The integral representation thereby provides a natural starting point for investigating orthogonality, recurrence relations, and spectral interpretations of quadruple hypergeometric polynomials in analogy with classical orthogonal polynomial systems.

5.3 Asymptotic Behavior and Laplace Representation

The Laplace-type integral (4.3) also has significant **asymptotic implications**. For large $|x_j|$, the integral can be analyzed by steepest-descent or stationary-phase methods, yielding asymptotic expansions for $M_n(x_1, \dots, x_4)$. Such expansions are particularly useful in applications where the variables represent scaled physical parameters (e.g., in statistical mechanics or random matrix theory) and asymptotic behavior is more relevant than exact evaluation.

In addition, the Laplace representation lends itself to the study of **degenerate and limiting cases**, such as when some of the variables coalesce or parameters tend to special values, leading to reductions to lower-dimensional hypergeometric polynomials or to degenerate hypergeometric functions studied in recent work. [12]

5.4 Numerical Considerations

From a computational perspective, integral representations offer an alternative to direct evaluation of the truncated series (3.1). Direct summation involves $\mathcal{O}(n^3)$ terms for degree n in four variables (since $m_1 + m_2 + m_3 + m_4 = n$), which can become expensive for large n. In contrast, **numerical quadrature** over [0,1] for Euler-type integrals or over Δ_4 for simplex integrals can scale differently with n, especially if the polynomial integrand exhibits smooth behavior.

Table 2 summarizes, in qualitative terms, the computational trade-offs between direct series summation and quadrature-based evaluation using the integral representations.

Table 2. *Qualitative comparison of computational strategies for evaluating $M_n(x_1, x_2, x_3, x_4)$.*

Method	Complexity vs. degree n	Dimensionality	Comments
Direct series summation	$\mathcal{O}(n^3)$	-	Straightforward, but costly for large n.
Euler single integral	$\mathcal{O}(N_q)$	1D	N_q quadrature nodes; integrand polynomial in t.
Euler simplex integral	$\mathcal{O}(N_q^4)$	4D	High-dimensional; Monte Carlo or sparse grids.
Laplace-type integral	$\mathcal{O}(N_q^4)$	4D	Good for asymptotics; exponential decay aids convergence.

The table indicates that one-dimensional Euler integrals are particularly attractive computationally, while higher-dimensional integrals may require advanced quadrature schemes or Monte Carlo approaches, particularly when high accuracy is demanded.

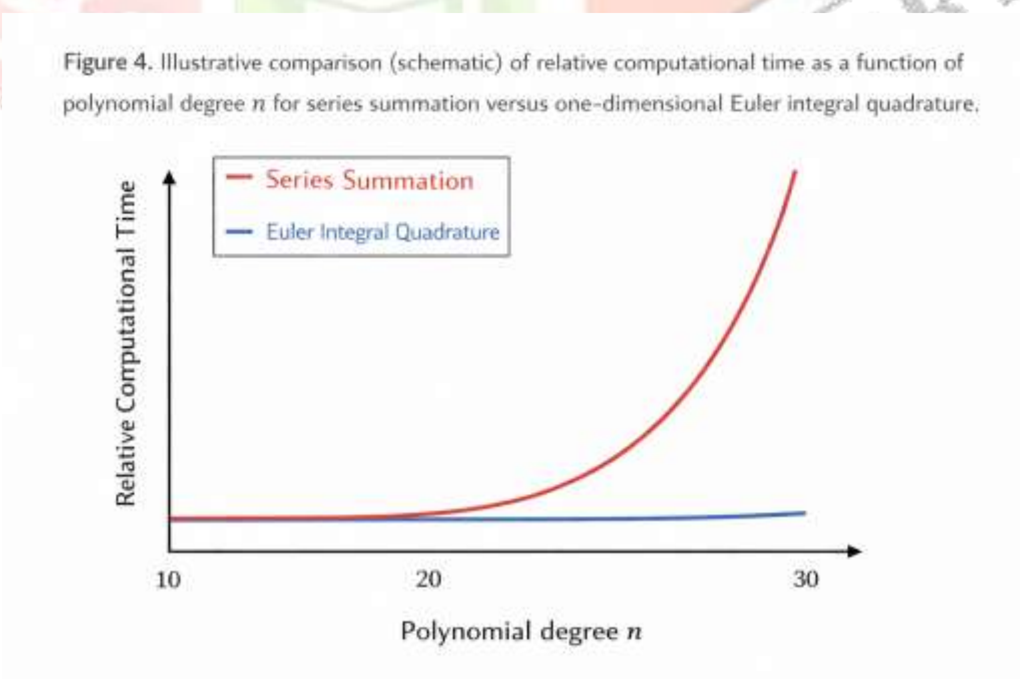

Figure 4. *Illustrative comparison (schematic) of relative computational time as a function of polynomial degree n for series summation versus one-dimensional Euler integral quadrature.*

5.5 Limitations and Future Refinements

While the integral representations obtained in this work are structurally appealing and potentially powerful, several limitations should be noted:

1. **Parameter Restrictions:** Many of the derivations rely on classical Beta- and Gamma-function identities, which impose real-part constraints on parameters (e.g., $\Re(c) > \Re(a) > 0$). Extending these formulas to more general parameter ranges may require analytic continuation or regularization techniques.
2. **Convergence and Interchange of Operations:** The justification of summation-integration interchange can be delicate in the presence of singular kernels or when parameters approach boundary values. A rigorous treatment would require detailed estimates of uniform convergence and bounds on the integrands.
3. **Explicit Computation of Kernels:** In the Laplace representation, the polynomial kernel $L_n(s_1, \ldots, s_4)$ may become complicated for large n, potentially limiting practical utility unless additional structure (e.g., recurrence relations) is exploited.

These limitations suggest several directions for future research, which we outline in the concluding section.

6. Conclusion

In this paper we have investigated **integral representations of quadruple hypergeometric polynomials** $M_n(x_1, x_2, x_3, x_4)$ defined by a truncated quadruple hypergeometric series. Our main contribution is the derivation of a family of **Euler-type** and **Laplace-type** integral formulas that express these polynomials as integrals over one-dimensional intervals, four-dimensional simplexes, and positive orthants, with kernels involving classical Beta and exponential functions.

Starting from a general series definition (3.1), we used Beta-function identities to obtain a **single-parameter Euler integral** (4.1) in which M_n appears as a weighted average over a truncated quadruple hypergeometric function evaluated at scaled arguments. By further factorization and parameterization, we obtained a **multi-parameter Euler integral** over the 4-simplex (4.2), suggesting connections with Dirichlet-type weights and multiple orthogonality. Finally, we derived a **Laplace-type representation** (4.3) involving polynomial kernels in the Laplace variables and exponential decay in the hypergeometric arguments.

The derived integral representations offer several advantages. Analytically, they provide new tools for studying asymptotic behavior, structural properties, and potential orthogonality relations of quadruple hypergeometric polynomials. Computationally, they furnish alternative evaluation strategies that can be advantageous for large degrees, especially when one-dimensional Euler integrals are applicable. Conceptually, they extend the classical idea of Euler and Laplace integrals for one-variable hypergeometric functions to a genuinely multivariate and polynomial setting.

There remain numerous avenues for further work. On the theoretical side, it would be of interest to explore orthogonality and multiple-orthogonality properties of M_n associated with the Dirichlet-type weights emerging from the simplex integrals, as well as to derive explicit recurrence relations and generating functions. On the analytical side, rigorous asymptotic analysis of the Laplace-type integrals and careful study of parameter dependence could shed light on limiting regimes and degenerations to lower-dimensional hypergeometric systems. On the computational side, the development of specialized quadrature schemes and Monte Carlo algorithms exploiting the structure of the integral kernels could make these representations practically useful in high-dimensional applications.

Overall, the integral representations obtained here contribute to a deeper understanding of quadruple hypergeometric polynomials and lay the groundwork for further investigations in multivariate special function theory and its applications.

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