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Generalized Fixed Point Theorems In Random Metric Space

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ABSTRACT

We prove some Common Fixed Point theorems for Random Operator in random metric spaces, by using some new type of contractive conditions taking non-self-mappings.

Key Words: - Random metric space, Random Operator, Random Multivalued Operator, Random Fixed, Point, Measurable Mapping, Non-self-mapping

AMS Subject Classification: - 47H10, 54H25.

1. Introduction

Random fixed point theorems represent a stochastic extension of classical fixed point theorems. Itoh [8] expanded upon several well-known fixed point theorems, and subsequently, various stochastic dimensions of Schauder's fixed point theorem have been explored by Sehgal and Singh [14], Papageorgiou [12], Lin [13], and numerous other authors. In a separable metric space, random fixed point theorems for contractive mappings were established by Spacek [15] and Hans [5,6]. Later, Beg and Shahzad [2], along with Badshah and Sayyad, examined the structure of common random fixed points and random coincidence points of a pair of compatible random operators, proving the random fixed point theorems for contraction random operators in random metric spaces.

2. Preliminaries: before starting main result we write some basic definetions.

Definition: 2.1:- A metric space (X, d) is said to be a Polish Space, if it satisfying following conditions:-

i. X, is complete,

ii. X is separable,

A metric space (X, d) is complete if whenever $(x_n: n \in \omega)$ is a sequence of member of X, such that for every $\epsilon > 0$ there is an N, such that $m, n \geq N$ implies

$$d(x_n, x_m) < \epsilon,$$

there is a single x in X such that $\lim_{n<\omega} x_n = x$.

It is easy to see that 2^{ω}, ω^{ω} are polish space, So in fact is ω under the discrete topology, whose metric is given by letting $d(x, y) = 1$ when $x \neq y$ and $d(x, y) = 0$ when $x = y$.

Let (X, d) be a Polish space that is a separable complete metric space and (Ω, q) be Measurable space.

Let 2^x be a family of all subsets of X and CB(X) denote the family of all nonempty bounded closed subsets of X.

A mapping $T: \Omega \to 2^X$ is called measurable if for any open subset C of X, $T^{-1}(C) = \{\omega \in \Omega: f(\omega) \cap C \neq \phi\} \in q$.

A mapping $\xi: \Omega \to X$ is said to be measurable selector of a measurable mapping $T: \Omega \to 2^X$, if ξ is measurable and for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega)$.

A mapping $f: \Omega \times X \to X$ is called random operator, if for any $x \in X$, $f(\cdot, x)$ is measurable.

A Mapping $T: \Omega \times X \to CB(X)$ is a random multivalued operator, if for every $x \in X$, $T(\cdot, x)$ is measurable.

A measurable mapping $\xi: \Omega \to X$ is called random fixed point of a random multivalued operator $T: \Omega \times X \to CB(X)$ $(f: \Omega \times X \to X)$ if for every $\omega \in \Omega$, $\xi(\omega) \in T\big(\omega, \xi(\omega)\big)$, $f(\omega), \xi(\omega) = \xi(\omega))$.

Let $T: \Omega \times X \to CB(X)$ be a random operator and $\{\xi_n\}$ a sequence of measurable mappings, $\xi_n: \Omega \to X$. Then sequence $\{\xi_n\}$ is said to be asymptotically T-regular if $d(\xi_n(\omega), T\big(\omega, \xi_n(w)\big) \to 0$.

3. Main Results

Theorem 3.1: Let X be a Random metric space. Let $T, S: \Omega \times X \to CB(X)$ be two continuous random multivalued operators. If there exists measurable mappings $\alpha, \beta, \gamma, \delta : \Omega \to (0,1)$ such that,

$$H\big(S(\omega,x),T(\omega,y)\big) \le \alpha(\omega)\frac{\max\{d^2(x,S(\omega,x)),d^2(y,T(\omega,y))\}}{d(x,y)} + \beta(\omega)\frac{\max\{d^2(y,S(\omega,x)),d^2(x,T(\omega,y))\}}{d(x,y)} \qquad 3.1(a)$$

For each $x, y \in X$, $\omega \in \Omega$ and $\alpha,\beta,\gamma,\delta \in R^+$ with $0 \le \alpha(\omega)+2\beta(\omega)+\gamma(\omega)+2\delta(\omega) < 1$,and $1-\beta(\omega) \neq 0$ there exists a common random fixed point of S and T.

Proof : Let $\xi_0 : \Omega \to X$ be an arbitrary measurable mapping and choose a measurable mapping $\xi_1 : \Omega \to X$ such that $\xi_1(\omega) \in S(\omega,\xi_0(\omega))$ for each $\omega \in \Omega$. then for each $\omega \in \Omega$.

$$H\Big(S\big(\omega,\xi_0(\omega)\big),T\big(\omega,\xi_1(\omega)\big)\Big) \le \alpha(\omega)\frac{\max\Big\{d^2\Big(\xi_0(\omega),S\big(\omega,\xi_0(\omega)\big)\Big),d^2\Big(\xi_1(\omega),T\big(\omega,\xi_1(\omega)\big)\Big)\Big\}}{d(\xi_0,\xi_1)} + \beta(\omega)\frac{\max\Big\{d^2\Big(\xi_1(\omega),S\big(\omega,\xi_0(\omega)\big)\Big),d^2\Big(\xi_0(\omega),T\big(\omega,\xi_1(\omega)\big)\Big)\Big\}}{d(\xi_0,\xi_1)}$$

Further there exists a measurable mapping $\xi_2 : \Omega \to X$ such that for all $\omega \in \Omega$, $\xi_2(\omega) \in T\big(\omega,\xi_1(\omega)\big)$ and

$$d\big(\xi_1(\omega),\xi_2(\omega)\big) \le \alpha(\omega)\frac{\max\{d^2(\xi_0(\omega),\xi_1(\omega)),d^2(\xi_1(\omega),\xi_2(\omega))\}}{d(\xi_1,\xi_2)} + \beta(\omega)\frac{\max\{d^2(\xi_1(\omega),\xi_1(\omega)),d^2(\xi_0(\omega),\xi_2(\omega))\}}{d(\xi_1,\xi_2)}$$

$$d\big(\xi_1(\omega),\xi_2(\omega)\big) \le \frac{\alpha(\omega)+\beta(\omega)}{1-\beta(\omega)} d\big(\xi_0(\omega),\xi_1(\omega)\big)$$

Let $k = \frac{\alpha(\omega)+\beta(\omega)}{1-\beta(\omega)}$

This gives

$$d\big(\xi_1(\omega),\xi_2(\omega)\big) \le k\, d\big(\xi_0(\omega),\xi_1(\omega)\big)$$

By Beg and Shahzad [2, lemma 2.3], we obtain a measurable mapping $\xi_3 : \Omega \to X$ such that for all $\omega \in \Omega$, $\xi_3(\omega) \in S\big(\omega,\xi_2(\omega)\big)$ and

$$d\big(\xi_2(\omega),\xi_3(\omega)\big) \le \alpha(\omega)\frac{\max\{d^2(\xi_1(\omega),\xi_2(\omega)),d^2(\xi_2(\omega),\xi_3(\omega))\}}{d(\xi_2,\xi_3)}$$

$$+\beta(\omega)\frac{\max\{d^2(\xi_2(\omega),\xi_2(\omega)),d^2(\xi_1(\omega),\xi_3(\omega))\}}{d(\xi_2,\xi_3)}$$

$$d(\xi_2(\omega),\xi_3(\omega)) \le k\, d(\xi_1(\omega),\xi_2(\omega)) \le k^2 d(\xi_0(\omega),\xi_1(\omega))$$

Similarly, proceeding the same way, by induction, we get a sequence of measurable mapping $\xi_n:\Omega \to X$ suct that for $n > 0$ and for any $\omega \in \Omega$,

$$\xi_{2n+1}(\omega) \in S(\omega,\xi_{2n}(\omega)) \text{ , and } \xi_{2n+2}(\omega) \in T(\omega,\xi_{2n+1}(\omega))$$

This gives,

$$d(\xi_n(\omega),\xi_{n+1}(\omega)) \le kd(\xi_{n-1}(\omega),\xi_n(\omega)) \le \cdots \ldots \ldots \ldots \le k^n d(\xi_0(\omega),\xi_1(\omega))$$

For any $m, n \in N$ such that $m > n$, also by using triangular inequality we have

$$d(\xi_n(\omega),\xi_m(\omega)) \le \frac{k^n}{1-k} d(\xi_0(\omega),\xi_1(\omega))$$

Which tends to zero as $n \to \infty$. It follows that $\{\xi_n(\omega)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$. It implies that $\xi_{2n+1}(\omega) \to \xi(\omega)$. Thus we have for any $\omega \in \Omega$,

$$d\left(\xi(\omega),S(\omega,\xi(\omega))\right) \le d(\xi(\omega),\xi_{2n+2}(\omega)) + d\left(\xi(\omega),S(\omega,\xi_{2n+2}(\omega))\right)$$

$$d\left(\xi(\omega),S(\omega,\xi(\omega))\right) \le d(\xi(\omega),\xi_{2n+2}(\omega)) + H\left(T(\omega,\xi_{2n+1}(\omega)),S(\omega,\xi_{2n+2}(\omega))\right)$$

Therefore,

$$d\left(\xi(\omega),S(\omega,\xi(\omega))\right) \le d(\xi(\omega),\xi_{2n+2}(\omega))$$

$$+\alpha(\omega)\frac{\max\left\{d^2\left(\xi_{2n+2}(\omega),S(\omega,\xi_{2n+2}(\omega))\right),d^2\left(\xi_{2n+1}(\omega),T(\omega,\xi_{2n+1}(\omega))\right)\right\}}{d(\xi_{2n+2},\xi_{2n+1})}$$

$$+\beta(\omega)\frac{\max\left\{d\left(\xi_{2n+1}(\omega),S(\omega,\xi_{2n+2}(\omega))\right),d\left(\xi_{2n+2}(\omega),T(\omega,\xi_{2n+1}(\omega))\right)\right\}}{d(\xi_{2n+2},\xi_{2n+1})}$$

Taking as $n \to \infty$, we have

$$d\left(\xi(\omega),S(\omega,\xi(\omega))\right) \le (\alpha(\omega) + \beta(\omega))\, d\left(\xi(\omega),S(\omega,\xi(\omega))\right)$$

Which contradiction, hence $\xi(\omega) = S(\omega,\xi(\omega))$ for al l$\omega \in \Omega$. Similarly, for any $\omega \in \Omega$,

$$d\left(\xi(\omega),S(\omega,\xi(\omega))\right) \le d(\xi(\omega),\xi_{2n+1}(\omega)) + H\left(S(\omega,\xi_{2n}(\omega)),T(\omega,\xi_{2n+1}(\omega))\right)$$

Hence $\xi(\omega) = T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

It is easy to see that, $\xi(\omega)$ is common fixed point for S and T in X.

Uniqueness:- Let us assume that, $\xi^*(\omega)$ is another fixed point of S and T in X, different from $\xi(\omega)$, then we have

$$d\big(\xi(\omega), \xi^*(\omega)\big) \leq d\Big(\xi(\omega), S\big(\omega, \xi_{2n}(\omega)\big)\Big) + H\Big(S\big(\omega, \xi_{2n}(\omega)\big), T\big(\omega, \xi_{2n+1}(\omega)\big)\Big)$$

$$+ d\Big(T\big(\omega, \xi_{2n+1}(\omega)\big), \xi^*(\omega)\Big)$$

By using 3.1(a) and $n \to \infty$ we have,

$$d\big(\xi(\omega), \xi^*(\omega)\big) \leq 0$$

Which contradiction,

So we have, $\xi(\omega)$ is unique common fixed point of S and T in X.

Corollary 3.2:- Let X be a Random metric space. Let $S^p, T^q : \Omega \times X \to CB(X)$ be two continuous random multivalued operators. If there exists measurable mappings $\alpha, \beta, \gamma, \delta : \Omega \to (0,1)$ such that,

$$H\big(S(\omega, x), T(\omega, y)\big) \leq \alpha(\omega) \max\{d\big(x, S(\omega, x)\big), d\big(y, T(\omega, y)\big)\}$$

$$+ \beta(\omega) \max\{d\big(y, S(\omega, x)\big), d\big(x, T(\omega, y)\big)\} \quad 3.2(a)$$

For each $x, y \in X$, $\omega \in \Omega$ and $\alpha, \beta, \gamma, \delta \in R^+$ with $0 \leq \alpha(\omega) + 2\beta(\omega) < 1$, and $1 - \beta(\omega) \neq 0$, there exists a common random fixed point of S and T.

Proof: From the theorem 3.1, it is immediate to see that, the corollary is true. If not then we choose a $\xi_0 : \Omega \to X$ be an arbitrary measurable mapping and choose a measurable mapping $\xi_1 : \Omega \to X$ such that $\xi_1(\omega) \in S(\omega, \xi_0(\omega))$ for each $\omega \in \Omega$. then for each $\omega \in \Omega$, and by using 3.2(a) the result is follows.

Now our next result is generalization of our previous theorem 3.1, in fact we prove the following theorem.

Theorem 3.3: Let X be a Random metric space. Let $T, S: \Omega \times X \to CB(X)$ be two continuous random multivalued operators. If there exists measurable mappings $\alpha, \beta, \gamma, \delta : \Omega \to (0,1)$ such that,

$$H\big(S(\omega, x), T(\omega, y)\big) \leq \alpha(\omega) \frac{\min\left\{\begin{matrix}\max\{d^2(x,S(\omega,x)), d^2(y,T(\omega,y))\}, \\ \max\{d^2(y,S(\omega,x)), d^2(x,T(\omega,y))\}\end{matrix}\right\}}{d(x,y)} \quad 3.3(a)$$

For each $x, y \in X$, $\omega \in \Omega$ and $\alpha \in R^+$ with $0 \leq \alpha(\omega) < 1$, there exists a common random fixed point of S and T.

Proof :- Let $\xi_0 : \Omega \to X$ be an arbitrary measurable mapping and choose a measurable mapping $\xi_1 : \Omega \to X$ such that $\xi_1(\omega) \in S(\omega, \xi_0(\omega))$ for each $\omega \in \Omega$. then for each $\omega \in \Omega$.

$$H\left(S(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))\right) \leq \alpha(\omega) \frac{\min\begin{Bmatrix}\max\left\{d^2\left(\xi_0(\omega),S(\omega,\xi_0(\omega))\right),d^2\left(\xi_1(\omega),T(\omega,\xi_1(\omega))\right)\right\},\\ \max\left\{d^2\left(\xi_1(\omega),S(\omega,\xi_0(\omega))\right),d^2\left(\xi_0(\omega),T(\omega,\xi_1(\omega))\right)\right\}\end{Bmatrix}}{d(\xi_0,\xi_1)}$$

Further there exists a measurable mapping $\xi_2 : \Omega \to X$ such that for all $\omega \in \Omega, \xi_2(\omega) \in T(\omega, \xi_1(\omega))$ and

$$d(\xi_1(\omega),\xi_2(\omega)) \leq \alpha(\omega) \frac{\min\begin{Bmatrix}\max\{d(\xi_0(\omega),\xi_1(\omega)),d(\xi_1(\omega),\xi_2(\omega))\},\\ \max\{d(\xi_1(\omega),\xi_1(\omega)),d(\xi_0(\omega),\xi_2(\omega))\}\end{Bmatrix}}{d(\xi_1,\xi_2)}$$

$$d(\xi_1(\omega),\xi_2(\omega)) \leq \alpha(\omega)\, d(\xi_0(\omega),\xi_1(\omega))$$

By Beg and Shahzad [2, lemma 2.3], we obtain a measurable mapping $\xi_3 : \Omega \to X$ such that for all $\omega \in \Omega$, $\xi_3(\omega) \in S(\omega, \xi_2(\omega))$ and by using 3.3 (a), we have

$$d(\xi_2(\omega),\xi_3(\omega)) \leq \alpha(\omega)\, d(\xi_1(\omega),\xi_2(\omega)) \leq (\alpha(\omega))^2 d(\xi_0(\omega),\xi_1(\omega))$$

Similarly, proceeding the same way, by induction, we get a sequence of measurable mapping $\xi_n : \Omega \to X$ suct that for $n > 0$ and for any $\omega \in \Omega$,

$$\xi_{2n+1}(\omega) \in S(\omega,\xi_{2n}(\omega)), \text{ and } \xi_{2n+2}(\omega) \in T(\omega,\xi_{2n+1}(\omega))$$

This gives, $d(\xi_n(\omega),\xi_{n+1}(\omega)) \leq \alpha(\omega)d(\xi_{n-1}(\omega),\xi_n(\omega)) \leq \cdots \ldots \ldots \ldots \leq (\alpha(\omega))^n d(\xi_0(\omega),\xi_1(\omega))$

For any $m, n \in N$ such that $m > n$, also by using triangular inequality we have

$$d(\xi_n(\omega),\xi_m(\omega)) \leq \frac{(\alpha(\omega))^n}{1-\alpha(\omega)} d(\xi_0(\omega),\xi_1(\omega))$$

Which tends to zero as $n \to \infty$. It follows that $\{\xi_n(\omega)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$. It implies that $\xi_{2n+1}(\omega) \to \xi(\omega)$. Thus we have for any $\omega \in \Omega$,

$$d\left(\xi(\omega),S(\omega,\xi(\omega))\right) \leq d(\xi(\omega),\xi_{2n+2}(\omega)) + d\left(\xi(\omega),S(\omega,\xi_{2n+2}(\omega))\right)$$

$$d\left(\xi(\omega), S(\omega, \xi(\omega))\right) \le d(\xi(\omega), \xi_{2n+2}(\omega)) + H\left(T(\omega, \xi_{2n+1}(\omega)), S(\omega, \xi_{2n+2}(\omega))\right)$$

Therefore, by using 3.3(a) we have

$$d\left(\xi(\omega), S(\omega, \xi(\omega))\right) \le \alpha(\omega)\, d\left(\xi(\omega), S(\omega, \xi(\omega))\right)$$

Which contradiction, hence $\xi(\omega) = S(\omega, \xi(\omega))$ for all $\omega \in \Omega$. Similarly, for any $\omega \in \Omega$,

$$d\left(\xi(\omega), S(\omega, \xi(\omega))\right) \le d(\xi(\omega), \xi_{2n+1}(\omega)) + H\left(S(\omega, \xi_{2n}(\omega)), T(\omega, \xi_{2n+1}(\omega))\right)$$

Hence $\xi(\omega) = T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

It is easy to see that, $\xi(\omega)$ is common fixed point for S and T in X.

Uniqueness :- Let us assume that, $\xi^*(\omega)$ is another fixed point of S and T in X, different from $\xi(\omega)$, then we have

$$d(\xi(\omega), \xi^*(\omega)) \le d\left(\xi(\omega), S(\omega, \xi_{2n}(\omega))\right) + H\left(S(\omega, \xi_{2n}(\omega)), T(\omega, \xi_{2n+1}(\omega))\right)$$
$$+ d\left(T(\omega, \xi_{2n+1}(\omega)), \xi^*(\omega)\right)$$

By using 3.3(a) and $n \to \infty$ we have,

$$d(\xi(\omega), \xi^*(\omega)) \le 0$$

Which contradiction, So we have, $\xi(\omega)$ is unique common fixed point of S and T in X.

Corollary 3.4:- Let X be a Random metric space. Let $S^p, T^q : \Omega \times X \to CB(X)$ be two continuous random multivalued operators. If there exists measurable mappings $\alpha, \beta, \gamma, \delta : \Omega \to (0,1)$ such that,

$$H(S(\omega,x), T(\omega,y)) \le \alpha(\omega) \min \begin{Bmatrix} \max\{d^2(x, S(\omega,x)), d^2(y, T(\omega,y))\}, \\ \max\{d^2(y, S(\omega,x)), d^2(x, T(\omega,y))\} \end{Bmatrix} \quad 3.4(a)$$

For each $x, y \in X$, $\omega \in \Omega$ and $\alpha, \in R^+$ with $0 \le \alpha(\omega) < 1$ there exists a common random fixed point of S and T.

Proof:- From the theorem 3.3, it is immediate to see that, the corollary is true. If not then we choose a $\xi_0 : \Omega \to X$ be an arbitrary measurable mapping and choose a measurable mapping $\xi_1 : \Omega \to X$ such that $\xi_1(\omega) \in S(\omega, \xi_0(\omega))$ for each $\omega \in \Omega$. then for each $\omega \in \Omega$, and by using 3.3(a) the result is follows.

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