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On Completely Regular $(2, n)$-Semirings

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Abstract: In this paper we introduce the notion of a completely regular $(2, n)$-semiring. Also, we give necessary and sufficient conditions for a $(2, n)$-semiring to be completely regular and characterized the results.

Key words: regular semigroup, $(2, n)$-semirings, idempotent elements, completely regular $(2, n)$-semiring

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I. Introduction

Algebraic polyadic structures are applied in many disciplines such as theoretical physics, computer sciences, coding theory, automata theory and other. The study of ordinary groups and semigroups is extended by several authors in some recently published papers to the situation where the group operation is n-ary rather than binary. The presence of these theories has inspired us to apply the analogous analysis of ordinary semirings to the situation in which the ring operations are n-ary. It seems to require some familiarity with the theory of n-groups. We provide some definitions and findings that will be applied in the follow-up. The publications listed in the references provide a thorough analysis.

The generalization of algebraic structures was in active research for a long time; Timm [11] in 1967 proposed commutative n-groups; later Crombez [3] in 1972 generalized rings and named it as (m, n)-rings. It was further studied by Crombez and Timm [4], Leeson and Butson [6, 7], and by Dudek [8]. Recently the generalization of algebraic structures is studied by Chaudhari and Nemade [X]. Sen, Maity, and Shum [9] have discussed in detail completely regular semiring. Also, Daddi and Pawar [5] introduced the structure of completely regular ternary semirings. Our main purpose in this paper is to introduce the notion of completely regular (2, n)-semiring and to obtain various characterizations of it. The $(2, n)$-semiring (R, f, g) (which is a generalization of the ordinary semiring (R, +, ×), where R is a set with binary operations + and ×), using f and g which are binary and n-ary operations, respectively.

2. Preliminaries:

In this section we review some definitions and results which will be used in later sections.

Definition 2.1. Let S be a non-empty set and $S^m = \{f(x_1, x_2, \dots, x_m) : x_i \in S, \forall\, i, 1 \le i \le m\}$
Then a function $f: S^m \to S$, is called an m-ary operation.

Definition 2.2. A non-empty set S with an m-ary operation f is called m-ary groupoid and it is denoted by (S, f).

Notations: If $x_1, x_2, \dots, x_m \in S$ then we write this as $x_1^m \in S$.

Definition 2.3. (1) Let $x_1, x_2, x_3 \in S$, then associativity with respect to binary operation f is defined as

$$f(x_1, f(x_2, x_3)) = f(f(x_1, x_2), x_3).$$

(2) Let $x_1, x_2, \dots, x_n, a_1, a_2 \in S$. Then n-ary operation g is distributive over binary operation f if,

$$g(x_1^i, f(a_1, a_2), x_{i+1}^n) = f\left(g(x_1^i, a_1, x_{i+1}^n), \quad g(x_1^i, a_2, x_{i+1}^n)\right)$$

Definition 2.4. A non-empty set S with binary operation f, called addition and n-ary operation g called multiplication is called $(\mathbf{2}, \boldsymbol{n})$**-semiring** if it satisfies the following conditions:

1) (S, f) is a commutative semigroup.
2) (S, g) is an n-ary semigroup.
3) there exists $0 \in S$ (called zero element of S) such that
   a) 0 is f-identity of S (i.e. $f(x, 0) = x$ for every $x \in S$.
   b) 0 is g-zero of S (i.e. $x_1, x_2, \dots, x_n \in S$ and $x_i = 0$ for some i implies that $g(x_1, x_2, \dots, x_n) = 0$.)
4) g is distributive over f.

From now onward, unless stated otherwise, S will denote a $(2, n)$-semiring.

Example 2.5.[2] The set of all non-positive integers $(\mathbb{Z}_0^-)$ is a $(2, 3)$-semiring (i.e. ternary semiring.

Example 2.6.[10] If $R = \{ni : n \in \mathbb{Z}_0^+ \ and \ i = \sqrt{-1}\}$, then R forms a $(2, 5)$-semiring.

Example 2.7.[1] Let $(R, f, g) = \{0, 2, 4, 6, \dots \dots\} = 2\mathbb{Z}_0^+$ where $f(a, b) = a + b$, and $g(a, b, c, d) = \frac{abcd}{8}$, then (R, f, g) forms a $(2, 4)$-semiring.

Definition 2.8. Let (S, f) be semigroup, then, $a \in S$ is called regular if there exists an element $x \in S$ such that $f(a, x, a) = a$.

Note: S is called regular if all its elements are regular.

3. Completely regular $(\mathbf{2}, \boldsymbol{n})$-semiring:

Definition 3.1. An element $a \in S$ is called completely regular if there exists $x \in S$ satisfying the following conditions:

i) $a = f(f(a, x), a)$

ii) $g(a^{n-1}, f(a, x)) = f(a, x)$

Definition 3.2. S is a completely regular $(2, n)$-semiring if every element of S is completely regular.

Example 3.3.[1] Let $(S, f, g) = \{0, 4, 8, 12, 16, 20\}$ where $f(a, b) = a + b \ (mod\ 24)$ and $g(a, b, c, d) = \frac{abcd}{64} \ (mod\ 24)$, then (S, f, g) is a completely regular $(2, n)$-semiring.

Theorem 3.4. S is a completely regular $(2, n)$-semiring if and only if for any $a \in S$ there exists $x \in S$ such that the following conditions are satisfied:

i) $a = f(f(a, x), a)$

ii)' $g(f(a, x), a^{n-1}) = f(a, x)$

Proof: Suppose that S is a completely regular $(2, n)$-semiring, let $a, x \in S$.

We have
$$f\big(g(a^n), g(f(a, x), a^{n-1})\big) = f\big(g(f(a, x), a^{n-1}), g(a^n)\big)$$
$$= g(f(f(a, x), a), a^{n-1})$$
$$= g(a, a^{n-1}) = g(a^n).$$

Hence,

$$f(\, g(a^{n-1}, x), f(a^n, g(f(a, x)), a^{n-1})\,) = f\big(g(a^{n-1}, x), g(a^n)\big)$$

This shows that $f(\, f(g(a^{n-1}, x), a^n), g(f(a, x), a^{n-1})\,) = g\big(a^{n-1}, f(x, a)\big)$.

Thus $f\left(g\big(a^{n-1}, f(x, a)\big), g(f(a, x), a^{n-1})\right) = g\big(a^{n-1}, f(x, a)\big)$.

Therefore, by second condition of definition of completely regular $(2, n)$-semiring,

$f\big(f(a, x), g(f(a, x), a^{n-1})\big) = f(a, x)$. ---------------------------------------(1)

Now, $f\big(g\big(a^{n-1}, f(x, a)\big), a^n\big) = g\big(a^{n-1}, f(f(a, x), a)\big) = g(a^n)$

So again, by second condition $f\big(f(a, x), g(a^n)\big) = g(a^n)$.

Therefore, $f\left(f\big(f(a, x), g(a^n)\big), g(x, a^{n-1})\right) = f\left(g(a^n), g\big(x, g(a^{n-1})\big)\right)$

So, $f\big(f(a,x), g(f(a,x), a^{n-1})\big) = g(f(a,x), a^{n-1})$.------------------------(2)

Hence by (1) and (2) $g(f(a,x), a^{n-1}) = f(a,x)$.

The converse part is similar. So we omit it.

Theorem 3.5. S is a completely regular $(2,n)$-semiring if and only if for any $a \in S$ there exists $x \in S$ such that the following conditions are satisfied:

i) $a = f(f(a,x), a)$

ii)' $g\big(a^i, f(a,x), a^{n-i-1}\big) = f(a,x), \quad for\ 1 \le i \le n-1.$

Proof: Suppose that S is a completely regular $(2,n)$-semiring, let $a, x \in S$.

Condition i) holds, also we have $g\big(a^{n-1}, f(a,x)\big) = f(a,x)$

Also, by theorem 3.4

Consider, $f\left(g(a^n), g\big(a^i, f(a,x),\ a^{n-i-1}\big)\right) = g\big(a^i, f\big(a, f(a,x)\big), a^{n-i-1}\big)$

$= g\big(a^i, a, a^{n-i-1}\big) = g(a^n)$.

Hence, $f(g(a^{n-1}, x), f\left(g(a^n), g\big(a^i, f(a,x), a^{n-i-1}\big)\right) = f\big(g(a^{n-1}, x), g(a^n)\big)$

Therefore $f(\ f(g(a^{n-1}, x), a^n), g(a^i, f(a,x), a^{n-i-1})) = g(a^{n-1}, f(x,a))$

So, $f(g\big(a^{n-1}, f(x,a)\big), g(a^i, f(a,x), a^{n-i-1})) = g(a^{n-1}, f(x,a))$

Therefore $f\left(f(a,x), g\big(a^i, f(a,x), a^{n-i-1}\big)\right) = f(a,x)$ ------------------------(3)

Since $f\left(g\big(a^{n-1}, f(a,x)\big), g(a^n)\right) = g\big(a^{n-1}, f(f(a,x), a)\big) = g(a^n)$.

Therefore $f\big(f(a,x), g(a^n)\big) = g(a^n)$.

Thus, $f\left(f\big(f(a,x), g(a^n)\big), g\big(a^i, x, a^{n-i-1}\big)\right) = f\left(g(a^n), g\big(a^i, x, a^{n-i-1}\big)\right)$

Therefore $f\left(f(a,x), g\big(a^i, f(a,x), a^{n-i-1}\big)\right) = g\big(a^i, f(a,x), a^{n-i-1}\big)$ ----------------(4)

By (3) and (4) $g\big(a^i, f(a,x), a^{n-i-1}\big) = f(a,x)$.

Converse can be proved in similar manner.

In the following theorem we characterize a completely regular $(2,n)$-semiring.

Theorem 3.6. S is completely regular $(2,n)$-semiring if and only if for each $a \in S$, there exists $x \in S$ satisfying following conditions:

i) $a = f(f(a,x), a)$;

ii) $g\big(a^{n-1}, f(a,x)\big) = f(a,x)$;

iii) $g(f(a,x), a^{n-1}) = f(a,x)$;

iv) $g\big(a^i, f(a,x), a^{n-i-1}\big) = f(a,x)$;

v) $f(a, g(f(a,x),\ a^{n-1})\ ) = a$;

vi) $f\big(a, g(a^{n-1}, f(a,x)\big) = a$;

vii) $f\left(a, g\big(a^i, f(a,x), a^{n-i-1}\big)\right) = a$;

viii) $g\big(a^{n-1}, f(a,x)\big) = g\big(a^i, f(a,x), a^{n-i-1}\big) = g(\ f(a,x), a^{n-1}\ )$.

Proof: Let S be regular $(2,n)$-semiring. Let $a \in S$, then by theorems 3.4 and 3.5, there exists $x \in S$ such that $a = f(f(a,x), a)$, $g\big(a^{n-1}, f(a,x)\big) = f(a,x)$, $g(f(a,x), a^{n-1}) = f(a,x)$ and $g\big(a^i, f(a,x), a^{n-i-1}\big) = f(a,x)$.

Also $f\big(a, g(f(a,x),\ a^{n-1})\big) = f\big(a, f(a,x)\big) = f\big(a, f(x,a)\big) = a$.

Similarly, we can show $f\big(a, g(a^{n-1}, f(a,x)\big) = a = f\left(a, g\big(a^i, f(a,x), a^{n-i-1}\big)\right)$.

Thus, all conditions are satisfied.

Converse part is obvious.

Definition 3.7. Let S be regular semigroup, let $a \in S$, then $x \in S$ is said to be inverse element of a, if $f(f(a,x), a) = a$ and $f\big(x, f(a,x)\big) = x$.

Theorem 3.8. Let S be a $(2,n)$-semiring and $a \in S$. Also let $V^+(a)$ denote the set of all inverse elements of a, if a is completely regular, then there exists a unique element $y \in V^+(a)$ such that, *i)* $g(a^{n-1}, f(a,y)) = f(a,y);$

ii) $f(a, g(f(a,y),\ a^{n-1})\ ) = a;$

iii) $f(g(a^{n-1}, f(a,y), a) = a;$

iv) $f(g(a^i, f(a,y), a^{n-i-1}), a) = a;$

v) $g(a^{n-1}, f(a,y)) = g(a^i, f(a,y), a^{n-i-1}) = g(\ f(a,y)\ , a^{n-1}\ ).$

Proof: Let $a \in S$ be completely regular element. Hence by theorem 3.6 there exists $x \in S$ such that $a = f(f(a,x), a),\ g(a^{n-1}, f(a,x)) = f(a,x),$

$$g(f(a,x), a^{n-1}) = f(a,x),\ g(a^i, f(a,x), a^{n-i-1}) = f(a,x)$$

$$f(a, g(f(a,x), a^{n-1})\ ) = a,\ f(a, g(a^{n-1}, f(a,x)) = a,$$

$$f\left(a, g(a^i, f(a,x), a^{n-i-1})\right) = a, \text{ and}$$

$$g(a^{n-1}, f(a,x)) = g(a^i, f(a,x), a^{n-i-1}) = g(\ f(a,x)\ , a^{n-1}\ ).$$

Let, $y = f(x, f(a,x))$.

Now $f(a, f(y,a)) = f\left(a, f(f(x, f(a,x)), a)\right) = f\left(f(a, f(x,a)), f(x,a)\right)$

$$= f(a, f(x,a)) = a.$$

Also, $f(y, f(a,y)) = f\left(f(x, f(a,x)), f\left(a, f(x, f(a,x))\right)\right)$

$$= f\left(x, f(f(a,x), a), f(x, f(a,x))\right)$$

$$= f\left(f(x,a), f(x, f(a,x))\right)$$

$$= f\left(f(x, f(a, f(x,a), x))\right)$$

$$= f(x, f(a,x)) = y.$$

Therefore y is the inverse element of a. Hence $y \in V^+(a)$.

So $f(f(a,y), a) = a$ and

$$g(a^{n-1}, f(a,y)) = g\left(a^{n-1}, f\left(a, f(x, f(a,x))\right)\right)$$

$$= g\left(a^{n-1}, f(f(a, f(x,a), x))\right)$$

$$= g(a^{n-1}, f(a,x))$$

$$= f(a,x)$$

$$= f(f(a, f(x,a)),\ x)$$

$$= f\left(a, f(x, f(a,x))\right) = f(a,y).$$

And $f(g(a^{n-1}, f(a,y)), a) = f(f(a,y), a) = a.$

Similarly, we can show $f(g(a^{n-1}, f(a,y), a) = a,\ f(g(a^i, f(a,y), a^{n-i-1}), a) = a,$ and $g(a^{n-1}, f(a,y)) = g(a^i, f(a,y), a^{n-i-1}) = g(\ f(a,y)\ , a^{n-1}\ ).$

Uniqueness: Let $z \in V^+(a)$ be another element satisfying the given conditions.

Hence, $y = f(f(y,a), y) = f(f(y,y), a)$

$$= f(f(y,y), f(f(a,z), a))$$

$$= f\left(f(y,y), f(f(a,z), f(f(a,z), a))\right)$$

$$= f\left(f(y,y), f\left(f(a, f(a,a), f(z,z))\right)\right)$$

$$= f\left(f\left(f(y, f(a,y)), f(a,a)\right), f(z,z)\right)$$

$$= f\left(f(y, f(a,a)), f(z,z)\right)$$

$$= f(f(f(a,y), a), f(z,z))$$

$$= f(a, f(z,z)) = f(f(z,a), z) = z$$

Theorem 3.9. If S is a completely regular $(2, n)$-semiring, then there exists unique element $y \in V^+(a)$ such that $g\left(a^{n-1}, f(a, y)\right) = f(a, y)$.

Proof: Straightforward by Theorem 3.8.

Theorem 3.10. If S is a completely regular $(2, n)$-semiring, then $E^+(S) = \{f(a, a'): a \in S\}$ and $g(e^n) = e$ for all $e \in E^+(S)$, where $E^+(S)$ is the set of all additive idempotents of S and a' is unique element as defined in Theorem 3.9.

Proof: Let $e \in E^+(S)$. Then there exists $x \in V^+(e)$ such that $f(f(e, x), e) = e$ and $g\left(e^{n-1}, f(e, x)\right) = f(e, x)$. Now $e = f(f(e, x), e) = f(e, x)$.

Also, $x = f(f(x, e), x) = f(x, e) = f(e, x) = e$.

Thus $E^+(S) = \{f(a, a'): a \in S\}$ and $g(e^n) = g(e^{n-1}, e) = g\left(e^{n-1}, f(e, x)\right) = f(e, x) = e$, for all $e \in E^+(S)$.

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