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# A Commutativity Theorem With Generalized Derivation On Jordan Ideal In Prime Rings

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***Abstract:*** **Let R be a 2-torsion free prime ring and $J$ be a non zero Jordan ideal of R. Suppose that $F$: R → R is a generalized derivation associated with non zero derivation $d$. If $F(xy) - d(x)F(y) - xy \in Z(\mathrm{R})$, for all $x, y \in J$, then R is a commutative ring.**

***Index Terms*** **- Prime ring, Generalized derivation, Jordan ideal.**

## I. Introduction

In this paper, R denotes an associative ring and center $Z(\mathrm{R})$ denotes centre of R. A ring R is said to be prime ring if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$. A ring $R$ is said to be 2-torsion free, if $2x = 0$ implies $x = 0$ for all $x \in \mathrm{R}$. The Jordan Product is denoted as $o$, which is defined on R as $xoy = xy + yx$, for all $x, y \in \mathrm{R}$ and Lie product of $x, y$ is denoted as $[x, y]$, which is defined as $[x, y] = xy - yx$, for all $x, y \in \mathrm{R}$. An additive subgroup $J$ of R is called a Jordan ideal of R if $uor \in J$, for all $u \in J$ and $r \in \mathrm{R}$. An additive mapping $d : \mathrm{R} \to \mathrm{R}$ is said to be a derivation, if $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathrm{R}$. Motivated by linear operator, Bresar introduced the concept of generalized derivation, which is the generalization of the derivation defined as follows. An additive mapping $F$ : $\mathrm{R} \to \mathrm{R}$ is said to be a generalized derivation if there exists a derivation $d : \mathrm{R} \to \mathrm{R}$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in \mathrm{R}$.

Firstly, E.C. Posner [11] proved pioneer results with derivation in prime rings and established connection between additive functions and structure of a ring. Many authors have extended Posner's results, further information can be found in [9],[10]. In this line, Ram Awtar [1] proved some theorems on special subset of ring known as Jordan ideal and Lie idea and also proved that, if $J$ is a Jordan ideal of R then, $4j^2r, 4rj^2, 4jrj \in J$, where $j \in J$, $r \in \mathrm{R}$. Further Zaidi [12] et al. proved a result which states that if $R$ is a ring and $J$ is a non zero Jordan ideal of $R$ then, $2J[\mathrm{R}, \mathrm{R}] \subseteq J$ and $2[\mathrm{R}, \mathrm{R}]J \subseteq J$. In 1991 M. Bresar [3] introduced the concept of generalized derivation in rings which is the generalization of derivation because every derivation is a generalized derivation but not conversely. In this sequence, Ashraf [2] et al. proved that if R is a prime ring which is 2 torsion free and $F$ is a generalized derivation associated with derivation $d$ on R. If $F$ satisfies any one of the following conditions: (i) $F(xy) - xy \in Z(\mathrm{R})$; (ii) $F(xy) - yx \in Z(\mathrm{R})$; (iii) $F(x)F(y) - xy \in Z(\mathrm{R})$; (iv) $F(x)F(y) - yx \in Z(\mathrm{R})$, for all $x, y \in I$, where $I$ is an ideal of R, then R is commutative.

Recently, Oukhtite L. [7] et al. proved that, if $R$R is a 2-torsion free prime ring, $J$ is a non zero ideal of R and $F$ satisfies any one of the following conditions:

$$(i) F(xy) - xy \in Z(\mathrm{R});$$
$$(ii) F(xy) - yx \in Z(\mathrm{R});$$
$$\text{(iii) } F(x)F(y) - xy \in Z(\mathrm{R});$$
$$\text{(iv) } F(x)F(y) - yx \in Z(\mathrm{R}),$$

for all $x, y \in J$, then R is a commutative ring.

Motivated by the results of Oukhtite L. [7], we prove our main theorem.

## Preliminary results

The following Lemma will be used in the proof of main results;

**Lemma 2.1.** [[8], Lemma 2.6] *If $J$ is a non zero Jordan ideal such that $aJb = 0$, then either $a = 0$ or $b = 0$.*

**Lemma 2.2.** [[7], Fact 3] *If $R$ is a noncommutative ring such that, $a[r, xy]b = 0$ for all $x, y \in J, r \in R$, then either $a = 0$ or $b = 0$.*

**Lemma 2.3.** [[7], Fact 6 ] *Let $i$ be a positive integer and set $J_0 = J$, then $J_i = \{x \in J_{i-1} \mid d(x) \in J_i\}$ is a nonzero Jordan ideal, moreover if $J \cap Z(R) \neq 0$, then $J_i \cap Z(R) \neq 0$.*

**Lemma 2.4.** [[10], Lemma 2.2 ] *If $d$ is a derivation of $R$ such that $d(x^2) = 0$ for all $x \in J$, then $d = 0$.*

**Lemma 2.5.** [[6], Remarks 2.1 ] *Let $R$ be a prime ring and $J$ be a non zero Jordan ideal of $R$. If $d$ is a derivation on $R$ such that $d^2(J) = 0$, then $d = 0$.*

We leave the proofs of the following easy Lemma to the readers.

**Lemma 2.6.** *Let $R$ be a prime ring and $J$ be a Jordan ideal of $R$. If $y^2 = 0$ for all $y \in J$, then $J = \{0\}$.*

**Lemma 2.7.** *Let $J$ be a nonzero Jordan ideal of $R$. Suppose that $d$, is a deriva- tion on $R$ such that, $d(x) = x$ for all $x \in J$, then $d = 0$.*

## Main Result

**Theorem 3.1.** *Let $R$ be a 2-torsion free prime ring and $J$ be a nonzero Jordan ideal of $R$. Suppose that $F : R \to R$ is a generalized derivation, associated with nonzero derivations $d$, such that $F(xy) - d(x)F(y) - xy \in Z(R)$, for all $x, y \in J$, then $R$ is commutative.*

*Proof.* First of all, we show that $J \cap Z(R) \neq 0$. On contrary if $J \cap Z(R) = 0$. We have,

$$F(xy) - d(x)F(y) - xy \in Z(R) \quad (1) \text{for all } x, y \in J.$$

Replacing $y$ by $4[r, uv]y$ in (1), where $u, v \in J, r \in R$, we get

$$4(F(x[r, uv]) - d(x)F([r, uv]) - x[r, uv])y + 4x[r, uv]d(y) - 4d(x)[r, uv]d(y) \in Z(R) \quad (2)$$

for all $u, v, x, y \in J$. As $4x[r, uv]d(y), 4d(x)[r, uv]d(y) \in J \ \forall\ x, y, u, v \in J_1$.
Therefore $4(F(x[r, uv]) - d(x)F([r, uv] - x[r, uv])y + 4x[r, uv]d(y) - 4d(x)[r, uv]d(y) \in J, \ \forall\ x, y, u, v \in J_1$. But $J \cap Z(R) = 0$, hence we get,

$$(F(x[r, uv]) - d(x)F([r, uv] - x[r, uv])y + x[r, uv]d(y) - d(x)[r, uv]d(y) = 0 \quad (3)$$ for all $x, y, u, v \in J_1, r \in R$. Replacing $y$ by $4yz^2$ in (3), where $z \in J_1$. we get

$$(x - d(x))[r, uv]yd(z^2) \quad (4)$$

for all $x, y, u, v \in J_1, r \in R$. Using lemma 2.2 we obtain either $x - d(x) = 0$ or $yd(z^2) = 0$. If $yd(z^2) = 0$, this implies $d(z^2) = 0 \ \forall\ z \in J_1$. Then by Lemma 2.4 $d = 0$, a contradiction. If $x -$

$d(x) = 0\ \forall\ x \in J_1$, then in application of Lemma 2.7, again we get $d = 0$, a contradiction. Therefore $J \cap Z(R) \neq 0$.

Replacing $y$ by $4yu^2$ in (1), where $u \in J$, we get

$$4(F(xy) - d(x)F(y) - xy)u^2 + 4xyd(u^2) - 4d(x)yd(u^2) \in Z(R) \qquad (5)$$

for all $x, y, u \in J$. Since $F(xy) - d(x)F(y) - xy \in Z(R)$, we get

$$[xyd(u^2), u^2] - [d(x)yd(u^2), u^2] = 0 \qquad (6)$$

Replacing $y$ by $4u^2y$ in (6), and subtracting from (6), we obtain

$$[xd(u^2)yd(u^2), u^2] = 0 \qquad (7)$$

Replacing $x$ by $4xu^2$ in (7), we get

$$[xu^2yd(u^2), u^2] - [d(x)u^2 \qquad (8)$$

for all $x, y, u \in J$. As $4d(u^2)yd(u^2)x = 4(d(u)ou)yd(u^2)x = 2(d(u)ou)o(yd(u^2)x)$ $+2[(d(u)ou), yd(u^2)x] \in J$ for all $x, y, u, \in J_1$, t hen replacing $x$ by $4d(u^2)yd(u^2)x$ in (8) we obtain

$$[d(u^2)yd(u^2), u^2]xd(u^2)yd(u^2) = 0 \qquad (9)$$

for all $x, y, u \in J_1$. Replacing $x$ by $4xu^2$ in (9), we get

$$[d(u^2)yd(u^2), u^2]xu^2d(u^2)yd(u^2) = 0 \qquad (10)$$

ng, (9) by $u^2$ from right side, and subtracting from (10),we obtain

$$[d(u^2)yd(u^2), u^2]J[d(u^2)yd(u^2), u^2] = 0 \qquad (11)$$

for all $x, y, u \in J_1$. In light of Lemma 2.1, we get $[d(u^2)yd(u^2), u^2] = 0$, for all $x, y, u \in J_1$. Therefore $d(u^2)yd(u^2)u^2 - u^2d(u^2)yd(u^2) = 0$. As $4yd(u^2)z = 2yd(u^2)oz + 2[yd(u^2), z] \in J$ then replacing $y$ by $2yd(u^2)z$ we get,

$$d(u^2)y[d(u^2), u^2]zd(u^2) = 0 \qquad (12)$$

for all $x, y, z, u \in J_1$. Again by application of Lemma 2.1, we get either $d(u^2) = 0$ or $[d(u^2), u^2] = 0$. If $d(u^2) = 0$ for all $u \in J_1$ , by Lemma 2.4, $d = 0$ a contradiction, therefore we get

$$[d(u^2), u^2] = 0 \qquad (13)$$

for all $u \in J_1$. Let $0 \neq t \in J_1 \cap Z(R)$ and replacing $u$ by $2rt$, where $r \in R$, we obtain

$$[d(r^2), r^2] = 0 \qquad (14)$$

for all $r \in R$. Therefore in application of the Theorem 3 of [4], we find that $[R, R]d(R) = 0$, hence $R$ is commutative. □

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