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Methods For Solving Ordinary Differential Equations Of Higher Order With Constant Coefficient

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Abstract

This article provides a detailed overview of the techniques and methods used to solve higher-order ordinary differential equations having coefficients that remain constant. The article begins by introducing the general form of higher order differential equations and explaining the concept of constant coefficients. Next, the article presents the characteristic of equation and its roots, which are used to determine the nature of the solutions. The article then goes on to discuss the three possible cases: distinct and real roots, complex conjugate roots, and repeated roots, and present the general solution for each case, including examples to illustrate the application of the method. The article concludes with a brief discussion of applications of higher-order differential equations in engineering and physics. The aim of this article is to provide a clear and concise guide for students and researchers interested in this important topic.

***Index Terms* -** Third-order differential equations, Constant coefficients, Homogeneous linear equations,Real and distinct roots, Complex conjugate roots, Repeated roots .

I. INTRODUCTION

Ordinary differential equations of second order, whose coefficients remain constant throughout the equation are a class of equations that frequently arise in many branches of mathematics and physics, making them of fundamental importance in understanding and predicting physical phenomena. These equations have a wide range of applications in areas such as mechanics, electromagnetism, and quantum mechanics . Therefore, finding the solutions to these equations is an essential task for many researchers and students in various fields of study. In this article, we aim to provide a comprehensive and detailed overview of the methods used to solve this class of equations. We begin by introducing the Standardized format of the second-order differential equation that has coefficients that remain constant throughout the equation, which is a homogeneous linear equation. We explain the meaning of constant coefficients and why they play an important role in solving these equations. We then present the characteristic equation and its roots, which provide information about the nature of the solutions. We discuss the three possible cases: real and distinct roots, complex conjugate roots, and repeated roots. For each case, we derive the general solution and provide examples to illustrate the application of the method . One of the strengths of this article is that it covers not only the mathematical aspects of solving these equations but also their physical interpretations and applications. We discuss the connection between the solutions and physical systems and how the solutions can be used to analyse and predict the behaviour of these systems. We provide examples from physics and engineering to illustrate the applications of differential equation that has coefficients that remain constant throughout the equation The techniques and methods presented in this article are fundamental to many fields of study, including mathematics, physics, and engineering. The article aims to provide a comprehensive guide for students and researchers who are interested in this topic, and it may be used as a reference for solving problems in related fields.

II. THIRD ORDER HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENT

Consider the third order homogeneous linear differential equation, where all the coefficients are constants with real values.In other words,we will only consider equations of the form

$$a_0(x)\frac{d^3y}{dx^3} + a_1(x)\frac{d^2y}{dx^2} + a_2(x)\frac{dy}{dx} + a_3(x)y = 0 \qquad (1)$$

ssTherefore,we will look solution of the above equation (1) in the form of $y = e^{mx}$ will choose the constant m such e^{mx} satisfies the equation. Assuming that is a solution for a certain value of m we can write:

$$\frac{dy}{dx} = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx}, \frac{d^3y}{dx^3} = m^3e^{mx}$$

Substituting in (1), we obtain

$$a_0(x)m^3e^{mx} + \mathrm{a}_1(x)m^2e^{mx} + \mathrm{a}_2(x)me^{mx} + a_3(x)e^{mx} = 0$$

$$e^{mx}\{a_0(x)m^3 + \mathrm{a}_1(x)m^2 + \mathrm{a}_2(x)m + a_3(x)\} = 0$$

Given that $e^{mx} \neq 0$, we can derive a polynomial equation in the variable m:

$$a_0(x)m^3 + \mathrm{a}_1(x)m^2 + \mathrm{a}_2(x)m = 0$$

(2)

The equation mentioned above referred to as the auxiliary equation or the characteristic equation of the given differential equation (1). While solving the auxiliary equation, the following three cases may arise

I) All the roots are distinct and real.

II) All the roots are real but some are repeating.

III) All the roots are imaginary.

2.1 Case I: Distinct Real Roots

If m_1, m_2, m_3 are different roots of equation (2) then $y = e^{m_1x}, y = e^{m_2x}, y = e^{m_3x}$ Are independent solutions of equation (1). Therefore the general solution of equation (1) is $y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}$ where $\mathrm{c}_1, \mathrm{c}_2$ are arbitrary constants.

$$y = c_1e^{-x} + c_2\, e^{2x} + c_3e^{3x} \text{ Where } \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3 \text{ are arbitrary constants.}$$

2.1 Example:

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

Hence,

$$(m-1)(m-2)(m-3) = 0$$

$$m = 1,2,3$$

The The roots are distinct and real . Thus e^x, e^{2x} and e^{3x} are the solutions to the equation are given, and we can express the general solution as:

$$y = c_1e^x + c_2\, e^{2x} + c_3e^{3x} \text{ Where } c_1, c_2, c_3 \text{ are arbitrary constants.}$$

2.2 Case II: Repeated Real Roots

If the auxiliary equation (2) has repeated real roots that are distinct, then the general solution of equation (1) can be expressed as:

$$y = c_1e^{mx} + c_2e^{mx} + c_3e^{mx}$$

2.2 Example:

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 0$$

The auxiliary equation is

$$m^3 - 3m^2 + 2m + 6 = 0$$

$$(m-2)(m-2)(m-2) = 0$$

$$m = 2,22$$

The roots are distinct and real and. Thus e^{2x}, e^{2x} and $e^{2x}$are the solutions to the equation are given, and we can express the general solution as:

$$y = c_1e^{2x} + c_2e^{2x} + c_3e^{2x} \text{ Where } c_1, c_2, c_3 \text{ are arbitrary constants.}$$

2.3 Case III: Conjugate Complex Roots :

Assuming the auxiliary equation has a non-repeated complex number root of the form $a + ib$, we can infer that $a - ib$ (the conjugate complex number) is also a non-repeated root since the coefficients in the equation are real. Therefore, the corresponding part of the general solution is:

$$y = c_1e^{mx} + c_2e^{(a+ib)x} + c_3e^{(a-ib)x} \text{ Where} c_1, c_2, c_3 \text{ are arbitrary constants.}$$

$$y = c_1e^{mx} + e^{ax}\left[c_2e^{ibx} + c_3e^{-ibx}\right]$$

$$= c_1e^{mx} + e^{ax}[c_2(cosbx + isinbx) + c_3(cosbx - isinbx)]$$

$$= c_1e^{mx} + e^{ax}[(c_2 + c_3)cosbx + (c_2 - c_3)isinbx]$$

$$= c_1e^{mx} + e^{ax}[Acosbx + Bisinbx]$$

2.3 Example:

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 4y = 0$$

The auxiliary equation is

$$m^3 - 4m^2 + 6m - 4 = 0$$

$$(m-2)(m^2 - 2m + 2) = 0$$

Solving it,we find

$$m = 2,1 - i, 1 + i$$

Express the solution in a general form is:

$$y = c_1e^{2x} + e^x(c_2 sinx + c_3 cosx) \quad \text{Where } c_1, c_2, c_3 \text{ are arbitrary constants.}$$

III. THIRD ORDER NON-HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENT

Let us consider the non-homogeneous Differential Equation

$$\boldsymbol{a_0(x)\frac{d^3y}{dx^3} + a_1(x)\frac{d^2y}{dx^2} + a_2(x)\frac{dy}{dx} + a_3(x)y = F(x)} \qquad (3)$$

Where the coefficients are constants but the non-homogeneous term F in general a non-constant function of x. The general solution of the above equation may be written, where is the general solution of the corresponding homogeneous equation (1) with F replaced by zero and is called the complementary function, and it is a solution that contains no arbitrary constant.On the other hand, any solution of equation (1) that does not contain arbitrary constants is known as a particular integral.

3.1 Case-1:

If $F(x) = x$, polynomial in x,then

$$y_p = \frac{1}{f(D)}X = [f(D)]^{-1}X$$

This can be applying binomial expansion $[f(D)]^{-1}$ and multiplying term by term.

Sometimes the expansions are made by using partial fraction.

3.1 Example :

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 6y = 4\text{x} + 5$$

The auxiliary equation

$$m^3 - 3m^2 + 2m + 6 = 0$$

$$(m+1)(m-2)(m-3) = 0$$

$$m = -1,2,3$$

The roots are distinct and real.Thus $\mathrm{e}^{-\mathrm{x}}$, $\mathrm{e}^{2\mathrm{x}}$ and $\mathrm{e}^{3\mathrm{x}}$ are the solutions that satisfy the equation and the solution that is not associated with any particular initial condition can be expressed as the complementary solution

$$y_c = c_1e^{-x} + c_2\, e^{2x} + c_3e^{3x} \quad \text{Where } \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3 \quad \text{are arbitrary constants.}$$

The particular solution is,

$$y_p = Ax + B$$

Where A, B are constants undetermined coefficient to be determined.Taking the derivative of the equation yields:

$$y_p' = A\,, y_p'' = 0 and\; y_p''' = 0$$

Substituting these in equation we obtain,

$$0 - 3(0) + 2A + 6(Ax + B) = 4x + 5$$

$$2A + 6Ax + 6B = 4x + 5$$

Equating the coefficient of x and constant term we obtain,

$$2A + 6B = 5 \text{ and } 6A = 4$$

Solving this we get,

$$A = \frac{2}{3} \quad \text{and} \quad B = \frac{11}{18}$$

Substituting these we obtain,

$$y_p = \frac{2}{3}x + \frac{11}{18}$$

An expression for the general solution can be formulated as:

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2\, e^{2x} + c_3 e^{3x} + \frac{2}{3}x + \frac{11}{18}$$,Where c_1, c_2, c_3 are arbitrary constants.

3.2 Case-2:

If $F(x) = e^{ax}$ is a constant,then $y_p = \frac{e^{ax}}{f(a)}$ provide $f(a) \neq 0$.

3.2 Example:

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = e^{4x}$$

The auxiliary equation is

$$m^3 - 3m^2 + 2m + 6 = 0$$

$$(m-2)(m-2)(m-2) = 0$$

$$m = 2{,}22$$

The roots are distinct and real.Thus e^{2x}, e^{2x} and e^{2x} thee solutions we have obtained are valid and it is possible to express the complementary solution in the following manner:

$$y_c = c_1 e^{2x} + c_2 e^{2x} + c_3 e^{2x}$$ Where c_1, c_2, c_3 are arbitrary constants.

The particular solution is,

$$y_p = Ae^{4x}$$

Differentiating the equation,we obtain

$$y_p' = 4Ae^{4x}$$

$$y_p'' = 16Ae^{4x}$$

$$y_p''' = 64Ae^{4x}$$

Substituting these we obtain,

$$64Ae^{4x} - 6(16Ae^{4x}) + 12(4Ae^{4x}) - 8Ae^{4x} = e^{4x}$$

$$64Ae^{4x} - 96Ae^{4x} + 48Ae^{4x} - 8Ae^{4x} = e^{4x}$$

$$8e^{4x} = e^{4x}$$

$$8A = 1$$

$$A = \frac{1}{8}$$

Substituting the value A and B we obtain,

$$y_p = e^{4x}\frac{1}{8}$$

An expression for the general solution can be formulated as:

$$y = y_c + y_p$$

Then $y = c_1e^{2x} + c_2e^{2x} + c_3e^{2x} + \frac{1}{8}e^{4x}$

3.3 Case-3:

We know that $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(D)}$ if $f(a) \neq 0$. But if $f(a) = 0$, this becomes infinite and our method fails.Now $f(a) = 0$ means that $D - a$ is a factor $f(D)$.

Therefore let $f(D) = (D - a)\varphi(D)$.So that $\varphi(a)$.

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D - a)\varphi(D)}e^{ax}$$

$$= \frac{1}{(D-a)\varphi(D)}e^{ax} \quad as\ \varphi(a) \neq 0$$

$$= \frac{1}{(D-a)\ \varphi(a)}e^{ax}$$

$$= \frac{1}{\varphi(a)}e^{ax}\int e^{-ax}\, e^{ax}dx$$

$$= \frac{1}{\varphi(a)}e^{ax}\int dx = \frac{xe^{ax}}{\varphi(a)}$$

Now differentiating both the sides of $f(D) = (D - a)\varphi(D)$ for $\varphi(a) \neq 0$ with respect to D.$f'(D) = (D - a)\varphi' + \varphi(D)$. Putting $D = a$,

$f'(a) = 0 + \varphi(a)$.It means $\varphi(a) = f'(a) = \frac{1}{\varphi(a)}e^{ax}\int dx = \frac{xe^{ax}}{\varphi(a)}$

Becomes $\frac{1}{f(D)}e^{ax} = \frac{xe^{ax}}{f'(a)}$ Or $x\frac{1}{f'(D)}e^{ax}$. Again if $f'(a) = 0$ and $f''(a) \neq 0$ then $D - a$ is factor repeated twice and applying the above result once again,we get $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(D)}e^{ax}$.

3.3 Example:

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = e^{3x}$$

The auxiliary equation is

$$m^3 - 4m^2 + 5m - 2 = 0$$

$$(m - 1)(m - 2)(m - 1) = 0$$

$$m = 1,2,3$$

The roots are real and distinct. Thus e^x, e^{2x} and e^{3x} the solutions we have obtained are valid, and it is possible to express the complementary solution in the following manner:

$y_c = (c_1 + c_2)e^x + c_3 e^{2x}$ Where c_1, c_2, c_3 are arbitrary constants.

The particular solution is,

$$y_p = Ae^{3x}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$y_p''' = 27Ae^{3x}$$

$$(27A - 36A + 15A - 2A)e^{3x} = e^{3x}$$

$$4A = 1$$

$$A = \frac{1}{4}$$

So the particular solution is

$$y_p = \frac{1}{4}e^{3x}$$

An expression for the general solution can be formulated as:

$y = y_c + y_p$, then $y = (c_1 + c_2)e^x + c_3 e^{2x} + \frac{1}{4}e^{3x}$

3.4 Case-4:

If $F(x) = sinx \ or \ cosx$. Then $\frac{1}{f(D)^2} sin\,ax = \frac{1}{f(-a^2)} sin\,ax$ and $\frac{1}{f(a^2)} cos\,ax = \frac{1}{f(-a^2)} cos\,ax$. Except when $f(-a^2) = 0$.

We know $sin\,ax = sin\,ax$.

$$D(sin\,ax) = a\,cos\,ax$$

$$D^2(sin\,ax) = -a^2\,sin\,ax$$

$$D^3(sin\,ax) = -a^3 cos\,ax$$

Similarly $(D^2)^n\,sin\,ax = (-a^2)^n\,sin\,ax$

Thus $f(D^2) sin\,ax = f(-a^2)\,sin\,ax$ operating by $\frac{1}{f(D^2)}$ on both sides, we get $\frac{1}{f(D)^2}\,f(D^2) sin\,ax = \frac{1}{f(D)^2} f(-a^2) sin\,ax$ or $f(-a^2) \neq 0$.

Dividing by $f(-a^2)$,we get $\frac{1}{f(D)^2} sin\,ax = \frac{1}{f(-a^2)} sin\,ax$, provide $f(-a^2) \neq 0$.Similarly $\frac{1}{f(D)^2} cos\,ax = \frac{1}{f(-a^2)} cos\,ax$.

3.4 Example:

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = cos\ 2x$$

The auxiliary equation is

$$m^3 + m^2 + 3m + 2 = 0$$

$$(m^2 + m + 2)(m + 1) = 0$$

$$m_1 = -1, m_2 = \frac{-1 \pm \sqrt{3i}}{2}$$

The roots are real and distinct. Thus e^{-x} and $e^{-\frac{-1\pm\sqrt{3i}}{2}x}$ the solutions we have obtained are valid, and it is possible to express the complementary solution in the following manner:

$$y_c = c_1e^{-x} + e^{\frac{-1}{2}x}(c_2cos(\frac{\sqrt{3}}{2}x) + c_3sin(\frac{\sqrt{3}}{2}x))$$ Where c_1,c_2, c_3 are arbitrary constants.

The particular solution is,

$$y_p = Acos\ 2x + Bsin\ 2x$$

Differentiating we obtain

$$y_p' = -2Asin\ 2x + 2Bcos\ 2x$$

$$y_p'' = -4Acos\ 2x - 4Bsin\ 2x$$

$$y_p''' = 8Asin2x - 2Bcos2x$$

Substituting these in equation we obtain,

$$(8Asin2x - 2Bcos2x) + 2(-4Acos\ 2x - 4Bsin\ 2x) + 3(-2Asin\ 2x + 2Bcos\ 2x) + Acos\ 2x + Bsin\ 2x = cos\ 2x$$

$$(-8B - 8A - 2A + B)cos2x + (8A - 8B - 6B + A)sin2x = cos2x$$

Equating the coefficient of $cos\ 2x, sin\ 2x$ and constant term we obtain,

$$-9A - 7B = 1, 9A - 14B = 0$$

Solving this we get

$$A = -\frac{2}{27}, \qquad B = -\frac{1}{21}$$

Substituting the value, we obtain

$$y_p = -\frac{2}{27}cos\ 2x - \frac{1}{21}sin\ 2x$$

An expression for the general solution can be formulated as:

$$y = y_c + y_p$$

$$y = c_1e^{-x} + e^{\frac{-1}{2}x}(c_2cos(\frac{\sqrt{3}}{2}x) + c_3sin(\frac{\sqrt{3}}{2}x)) - \frac{2}{27}cos\ 2x - \frac{1}{21}sin\ 2x$$

Where c_1,c_2, c_3 are arbitrary constants.

REFERENCES

[1]. Adkins, W.A., & Davidson, M.G. (2012). Second Order Constant Coefficient Linear Differential Equations. In *Ordinary Differential Equations* (pp. 203–273). Springer.Ravi P. Agarwal"Ordinary Differential Equations" □□

[2]. Beged-Dov**,** A. G. (1966)**.** Solution of Higher-Order Linear Differential Equations with Constant Coefficients. *IEEE Transactions on Education*, 9, 38.

[3]. .J. E. Boyce and R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems,11th ed. Hoboken, NJ: Wiley, 2017.

[4]. Chaundy, T. W., & Launchbury, H. R. (1937). Systems of ordinary differential equations with constant coefficients. *The Quarterly Journal of Mathematics*, os-8(1), 214–218.

[5]. Churchill, R. V., & Brown, J. W. (1990). Complex variables and applications (4th ed.). New York: McGraw-Hill.

[6]. Coddington, E. A. (1955)**.** An Introduction to Ordinary Differential Equations. Prentice-Hall.

[7]. Coddington, E. A. (1961). Theory of Ordinary Differential Equations. McGraw-Hill.

[8]. Coddington, E. A. (1965). Contributions to the Theory of Nonlinear Oscillations. Proceedings of the

[9]. Euler, L. (1748). Institutiones calculi differentialis. St. Petersburg.

[10]. Shohal Hossian, Samme Akter Mithy, Methods for solving ordinary differential equations of second order with coefficients that is constant