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Rigorous Approach To The Stability Of Cubic Functional Equations In Quasi-Beta Banach Space

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Abstract: In this pioneering research, we leverage the direct and fixed-point approaches to meticulously prove the Ulam-Hyers-Rassias stability of a cubic functional equation characterized by a complex and innovative mean sum formulation of successive function variables of the form

$$\sum_{k=1}^{n}\Gamma\left(\frac{1}{k}\sum_{l=1}^{k}\theta_l\right)=\sum_{k=1}^{n}\frac{1}{k^3}\left\{\left(\frac{(k-3)(k-2)}{2}\right)\sum_{l=1}^{n}\Gamma(\theta_l)+\sum_{1\le k<l<m}^{n}\Gamma(\theta_k+\theta_l+\theta_m)-(k-3)\sum_{1\le k<l}^{n}\Gamma(\theta_k+\theta_l)\right\}$$

in quasi-β-Banach space.

Key Words**—** Cubic functional equation, UHR Stability, quasi- β - Banach space, Complete Normed Space, Fixed point, Hyers Approach, Contractive Mapping.

I. Introduction

S.M. Ulam's [23] visionary concept of functional equation stability, introduced in 1964, posed a profound and intriguing question that has since become a cornerstone of mathematical inquiry: 'Can a function that precisely satisfies a functional equation be found in close proximity to another function that only approximately satisfies it?' This seminal query ignited a wave of intense research, and in 1941, D.H.Hyers [10] achieved a groundbreaking breakthrough, providing a pivotal partial solution that has had far-reaching and profound implications. A paradigm-shifting moment occurred in the realm of functional equation stability when T.Aoki [2] boldly expanded upon Hyers' seminal work, pushing the boundaries to additive mappings. However, it was Th.M.Rassias' [18] revolutionary and monumental generalization of Hyers' conclusion in 1978 that catapulted the field into a new era of intense scrutiny, exploration, and innovation. This groundbreaking achievement not only revitalized interest in the subject but also laid the foundation for the Ulam-Hyers-Rassias stability theory, which has since become a seminal and indispensable touchstone of functional equation research. For those seeking a deeper understanding of this dynamic, rapidly evolving, and increasingly complex field, a plethora of insightful and authoritative resources can be found in [3,4,5,6,7,8,10,13,14,15].

The main purpose of this research paper is to provide a comprehensive examination of the generalized UHR stability of cubic functional equations of the form

$$\sum_{k=1}^{n}\Gamma\left(\frac{1}{k}\sum_{l=1}^{k}\theta_l\right)=\sum_{k=1}^{n}\frac{1}{k^3}\left\{\left(\frac{(k-3)(k-2)}{2}\right)\sum_{l=1}^{n}\Gamma(\theta_l)+\sum_{1\le k<l<m}^{n}\Gamma(\theta_k+\theta_l+\theta_m)-(k-3)\sum_{1\le k<l}^{n}\Gamma(\theta_k+\theta_l)\right\}$$

(1)

in quasi- β - Banach space leveraging direct and fixed-point method.

We now review the basic outcome of alternative fixed-point theory.

Theorem. I.1. [The alternative fixed point] Suppose for a complete generalized metric space $(\psi, d)$and strictly contractive mapping $\xi : \psi \to \psi$ with Lipchitz constant L. Then given an arbitrary $\theta \in \psi$,either $d(\xi^n\theta, \xi^{n+1}\theta) = \infty$ for all $n \geq 0$,

Or there exist a natural number n_0 such that

(ABF1) $d(\xi^n\theta, \xi^{n+1}\theta) < \infty$ for all $n \geq n_o$;

(ABF2) The sequence $\{\xi^n\theta\}$ is converges strongly to a fixed point θ^* of ξ;

(ABF3) The fixed point θ^* is uniquely determined in the set of all points of ξ in the set $\varpi = \{\theta \in \psi : d(\theta, \xi^{n_o}\theta) < \infty\}$;

(ABF4) $d(\theta, \theta^*) \leq \frac{1}{1-L} d(\theta, \xi\theta)$ for every $\theta \in \psi$.

I.1 **BASICS OF QUASI-BETA BANACH SPACE.**

In this section, first we present here some basic facts in [24,25] concerning quasi-β-Normed space and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let λ denote either R or C.

Definition I.2. Let K be a linear space over F . A quasi-β-norm $\|.\|$ is a real-valued function on K satisfying the following:

(i) $\| k \| \geq 0$ for all $k \in K$ and $\| k \| = 0$ if and only if $k = 0$.

(ii) $\| \lambda k \| = \lambda^\beta \| k \|$ for all $\lambda \in F$ and all $k \in K$.

(iii) There is a constant $\lambda \geq 1$ such that $\| k + t \| \leq \lambda(\| k \| + \| t \|)$ for all $k, t \in K$.

The pair $(K, \|.\|)$ is called quasi-β-normed space if $\|.\|$ is a quasi-β-norm on K. The smallest possible λ is called the modulus of concavity of $\|.\|$.

Definition I.3. A quasi-β-Banach space is a complete quasi-β-normed space.

Now, the authors presented the generalized Ulam - Hyers stability of the functional equation (1) in quasi-Beta normed space.

II. STABILITY RESULT IN QUASI BETA BANACH SPACE VIA DIRECT METHOD

In this section, we employ the direct method to scrutinize the Ulam-Hyers-Rassias stability of functional equation (1) within the framework quasi-β- Banach space.

Herein after, unless otherwise specified we assume that ψ_1 be a normed linear space and ψ_2 be a quasi-β- Banach space.

For a given function $\Gamma : \psi_1 \to \psi_2$, we adopt the following notation

$$\partial\Gamma(\theta_1, \theta_2, \ldots, \theta_n) = \sum_{k=1}^{n} \Gamma\left(\frac{1}{k}\sum_{l=1}^{k}\theta_l\right) - \sum_{k=1}^{n}\frac{1}{k^3}\left\{\begin{aligned}&\left(\frac{(k-3)(k-2)}{2}\right)\sum_{l=1}^{n}\Gamma(\theta_l)\\ &+\sum_{1\leq k<l<m}^{n}\Gamma(\theta_k + \theta_l + \theta_m) - (k-3)\sum_{1\leq k<l}^{n}\Gamma(\theta_k + \theta_l)\end{aligned}\right\}$$

for all $\theta_i \in \psi_1, (i = 1,2,3,\ldots,n)$.

Ulam-Hyer's Stability Analyze. To analyze the stability result, we establish the following:

- ✓ **Existence of Solution**:
  We demonstrate the existence of a solution to the functional equation.
- ✓ **Satisfaction of Solution:**
  We verify that the solution satisfies the functional equation.
- ✓ **Uniqueness of Solution:**
  We prove that the solution is unique.

Theorem II.1 Let $\tau = \pm 1$ be fixed. Also let $\lambda : \psi_1{}^n \to [0, \infty)$ be a mapping fulfills

$$\lim_{s\to\infty}\frac{\lambda\left(n^{\tau s}\theta_1,n^{\tau s}\theta_2,\ldots,n^{\tau s}\theta_n\right)}{n^{3s\tau}}=0$$

(2)

for every $\theta_i\in\psi_1,(i=1,2,3,\ldots,n)$ and $\Gamma:\psi_1\to\psi_2$ be a cubic operator obey the inequality

$$\left\|\partial\Gamma(\theta_1,\theta_2,\ldots,\theta_n)\right\|\le\lambda(\theta_1,\theta_2,\ldots,\theta_n)$$

(3)

for each $\theta_i\in\psi_1,(i=1,2,3,\ldots,n)$. Then there exists a unique cubic correspondence $\mathrm{H}:\psi_1\to\psi_2$ such that

$$\left\|\Gamma(\theta)-\mathrm{H}(\theta)\right\|\le\sum_{k=\frac{1-\tau}{2}}^{\infty}\frac{\alpha\left(n^{\tau k}\theta\right)}{n^{3\beta k\tau}}$$

(4)

for all $\theta\in\psi_1$, where the correspondence $\mathrm{H}(\theta)$ and $\alpha(\theta)$ are defined by $\mathrm{H}(\theta)=\lim_{s\to\infty}\frac{\Gamma\left(n^{\tau s}\theta\right)}{n^{3k\tau}}$ and $\alpha(\theta)=\lambda(0,0,\ldots,n\theta)$ for every $\theta\in\psi_1$ respectively.

Proof. **Case (i)**: Assume $\tau=1$. Replacing $(\theta_1,\theta_2,\ldots,\theta_n)$ by $(0,0,\ldots,\theta)$ in (3), we realize

$$\left\|\Gamma\left(\frac{\theta}{n}\right)-\frac{\Gamma(\theta)}{n^3}\right\|\le\lambda(0,0,\ldots,\theta)$$

(5)

for all $\theta\in\psi_1$. Changing θ by $n\theta$ in (5), we obtain

$$\left\|\Gamma(\theta)-\frac{\Gamma(n\theta)}{n^3}\right\|\le\lambda(0,0,\ldots,n\theta)$$

(6)

for all $\theta\in\psi_1$.Taking $\lambda(0,0,\ldots,n\theta)=\alpha(\theta)$ into (6), we arrive

$$\left\|\Gamma(\theta)-\frac{\Gamma(n\theta)}{n^3}\right\|\le\alpha(\theta)$$

(7)

for all $\theta\in\psi_1$. In general

$$\left\|\Gamma(\theta)-\frac{\Gamma\left(n^s\theta\right)}{n^{3s}}\right\|\le\lambda^{s-1}\sum_{k=0}^{s-1}\frac{\alpha\left(n^k\theta\right)}{n^{3\beta k}}$$

(8)

for all $\theta\in\psi_1$. Setting θ by $n^l\theta$ in (8) and dividing $n^{3l\beta}$, we land

$$\left\|\frac{\Gamma\left(n^l\theta\right)}{n^{3l}}-\frac{\Gamma\left(n^{s+l}\theta\right)}{n^{3(s+l)}}\right\|\le\lambda^{s-1}\sum_{k=0}^{s-1}\frac{\alpha\left(n^{k+l}\theta\right)}{n^{3\beta(k+l)}}$$

(9)

for all $\theta\in\psi_1$. As limit $l\to\infty$ the right off (9) approaches to 0. It follows that the sequence $\left\{\frac{\Gamma(n^s\theta)}{n^{3s}}\right\}$ is a Cauchy sequence in ψ_2. Since ψ_2 is Banach, therefore the sequence $\left\{\frac{\Gamma(n^s\theta)}{n^{3s}}\right\}$ reaches a limiting point $\mathrm{H}(\theta)\in\psi_2$. So, we define

$$\mathrm{H}(\theta)=\lim_{s\to\infty}\frac{\Gamma\left(n^s\theta\right)}{n^{3s}}$$

(10)

for all $\theta\in\psi_1$. To demonstrate that H meets (1), replacing $(\theta_1,\theta_2,\ldots,\theta_n)$ by $\left(n^s\theta_1,n^s\theta_2,\ldots,n^s\theta_n\right)$ and dividing n^{3s} in (3) and allowing limit $s\to\infty$ and using (10), a simple observation reveals that that H satisfies (1) for every $\theta_i\in\psi_1,(i=1,2,3,\ldots,n)$. To prove H is idiosyncratic, we let H' be another mapping satisfies (1) and (3). These yields

$$\begin{aligned}\left\|\mathrm{H}(\theta)-\mathrm{H}'(\theta)\right\| &=\frac{1}{n^{3s}}\left\|\mathrm{H}(n^s\theta)-\mathrm{H}'(n^s\theta)\right\|\\ &\le\frac{\lambda}{n^{3s}}\left(\left\|\mathrm{H}(n^s\theta)-\Gamma(n^s\theta)\right\|+\left\|\Gamma(n^s\theta)-\mathrm{H}'(n^s\theta)\right\|\right)\end{aligned}$$

$$\leq 2\,\frac{\lambda^{s}}{n^{3s}}\sum_{k=0}^{\infty}\frac{\alpha\left(n^{k+s}\theta\right)}{n^{3\beta k}}$$

$$\to 0 \text{ as } s \to 0$$

for all $\theta \in \psi_1$. We thus conclude $\mathrm{H}(\theta)=\mathrm{H}'(\theta)$, this confirms the uniqueness of the solution.

Case (ii): For $\tau=-1$. Putting θ by $\frac{\theta}{n}$ in (7) and multiplying $n^{3\beta}$, we reach

$$\left\| n^{3}\Gamma\left(\frac{\theta}{n}\right)-\Gamma(\theta)\right\| \leq n^{3\beta}\alpha\left(\frac{\theta}{n}\right) \tag{11}$$

for all $\theta \in \psi_1$. In general

$$\left\| n^{3s}\Gamma\left(\frac{\theta}{n^{s}}\right)-\Gamma(\theta)\right\| \leq \lambda^{s-1}\sum_{k=1}^{s-1} n^{3\beta k}\alpha\left(\frac{\theta}{n^{k}}\right) \tag{12}$$

for all $\theta \in \psi_1$. The remaining argument is a direct analogue of **Case(i).** This concludes the rigorous proof of the Theorem.

As a direct implication of the Theorem II.1, we have the following corollary to concerning the stability of (1).

Corollary II.2 Let's suppose that ε is a positive number and ω is a real number with $\omega \neq 3$. Let $\Gamma:\psi_1 \to \psi_2$ be a cubic function fulfills the functional disproportion

$$\left\|\partial\Gamma(\theta_1,\theta_2,\ldots,\theta_n)\right\| \leq \begin{cases} \varepsilon; \\ \varepsilon\sum_{i=1}^{n}\|\theta_i\|^{\omega};\ \omega\neq 3 \end{cases} \tag{13}$$

for all $\theta_i \in \psi_1, (i=1,2,3,\ldots,n)$. Then there exists unique cubic correspondence $\mathrm{H}:\psi_1 \to \psi_2$ in such a way that

$$\|\Gamma(\theta)-\mathrm{H}(\theta)\| \leq \begin{cases} \dfrac{\lambda^{s-1}\varepsilon n^{3\beta}}{\left|n^{3\beta}-1\right|}; \\ \dfrac{\lambda^{s-1}\varepsilon n^{\beta(3+\omega)}\|\theta\|^{\omega}}{\left|n^{\omega\beta}-n^{3\beta}\right|};\ \omega\neq 3 \end{cases} \tag{14}$$

for all $\theta \in \psi_1$.

III. Stability Result in Quasi Beta Banach Space Using Fixed Point Method

In this section, we investigate the Ulam-Hyers-Rassias stability of the functional equation (1) within the Scaffolding of quasi -β- Banach space, employing the fixed-point method to establish our results.

Theorem III.1 Suppose $\Gamma:\psi_1 \to \psi_2$ be a mapping satisfies the disproportion (3) with respect to there exists a correspondence $\lambda:\psi_1{}^{n} \to [0,\infty)$ under the proviso that

$$\lim_{s\to\infty}\frac{\lambda\left(v_i{}^{s}\theta_1, v_i{}^{s}\theta_2,\ldots,v_i{}^{s}\theta_n\right)}{v_i{}^{3s}}=0, \text{ where } v_i=\begin{cases}\frac{1}{n}, i=0\\ n, i=1\end{cases} \tag{15}$$

for all $\theta_i \in \psi_1, (i=1,2,3,\ldots,n)$. In the presence of $L=L(i)$ such that the function $\theta \to \alpha(\theta)$ meets the specified conditions

$$\alpha(\theta)=\lambda(0,0,\ldots,n\theta);\quad L\alpha(\theta)=\frac{1}{v_i^{3\beta}}\alpha(v_i\theta) \tag{16}$$

for every $\theta \in \psi_1$.Then there exists unique cubic correspondence $\mathrm{H}:\psi_1 \to \psi_2$ fulfills the functional equation (1) and the disproportion

$$\|\Gamma(\theta)-\mathrm{H}(\theta)\| \leq \frac{L^{1-i}}{1-L}\alpha(\theta) \tag{17}$$

for all $\theta \in \psi_1$.

Proof. Postulate a set

$$\Phi = \{\Gamma : \Gamma : \psi_1 \to \psi_2, \Gamma(0) = 0\}$$

(18)

and define the generalized metric on Φ as

$$d(\Gamma, \Gamma') = \inf\{\delta > 0; \|\Gamma(\theta) - \Gamma'(\theta)\| \le \delta\alpha(\theta)\}$$

(19)

for all $\theta \in \psi_1$, it can be confirmed at a glance that (Φ, d) is complete.

Define a transformation $\Gamma : \Phi \to \Phi$ by

$$\xi_\Gamma(\theta) = \frac{\Gamma(\nu_i \theta)}{{\nu_i}^{3\beta}}$$

(20)

for all $\theta \in \psi_1$.We now declare ξ_Γ is contraction on Φ, if for any $\Gamma, \Gamma' \in \Phi$ subject to

$$d(\Gamma, \Gamma') \le \delta$$

$$\Rightarrow \qquad \|\Gamma(\theta) - \Gamma'(\theta)\| \le \delta\alpha(\theta)$$

$$\Rightarrow \qquad \left\| \frac{\Gamma(\theta\nu_i)}{{\nu_i}^3} - \frac{\Gamma'(\theta\nu_i)}{{\nu_i}^3} \right\| \le \delta \frac{\alpha(\theta\nu_i)}{{\nu_i}^{3\beta}}$$

$$\Rightarrow \qquad \|\xi_\Gamma(\theta) - \xi_{\Gamma'}(\theta)\| \le \delta\ L\ \alpha(\theta)$$

$$\Rightarrow \qquad d(\xi_\Gamma, \xi_{\Gamma'}) \le \delta L$$

$$\Rightarrow \qquad d(\xi_\Gamma, \xi_{\Gamma'}) \le L d(\Gamma, \Gamma')$$

(21)

this reveals that the mapping $\xi_\Gamma : \Phi \to \Phi$ is contractive mapping on Φ having a Lipchitz constant L. It's trail around with (7), we settle on

$$\left\| \Gamma(\theta) - \frac{\Gamma(n\theta)}{n^3} \right\| \le \alpha(\theta)$$

(22)

for all $\theta \in \psi_1$. With help of (19), (20) and (22) it is apparent that

$$d(\xi_\Gamma, \Gamma) \le 1 \text{ , for } \quad i = 1$$

$$\Rightarrow \qquad d(\xi_\Gamma, \Gamma) \le L^{1-i}$$

(23)

Similarly, from (11) for the case $i = 0$, it is apparent that

$$\left\| n^3 \Gamma\left(\frac{\theta}{n}\right) - \Gamma(\theta) \right\| \le L\alpha(\theta)$$

(24)

for all $\theta \in \psi_1$,where $L\alpha(\theta) = n^3 \alpha\left(\frac{\theta}{n}\right)$. With help of (19) ,(20) and (24), we end up at

$$d(\xi_\Gamma, \Gamma) \le L \text{ , for } \quad i = 0$$

$$\Rightarrow \qquad d(\xi_\Gamma, \Gamma) \le L^{1-i}$$

(25)

Combining (23) and (25), we come to

$$d(\xi_\Gamma, \Gamma) \le L^{1-i} < \infty$$

(26)

Therefore (ABF1) of Theorem I.1 holds.

By (ABF2) of Theorem I.1, there exists a fixed point H of ξ_Γ in Φ on the condition that

$$\mathrm{H}(\theta) = \lim_{n \to \infty} \frac{\Gamma({\nu_i}^n \theta)}{{\nu_i}^{3n}} \text{ , for all } \theta \in \psi_1.$$

It remains to show that $\mathrm{H} : \psi_1 \to \psi_2$ fulfilling the functional equation (1), the proof of this is analogous to the concept presented in Theorem II.1.

Again by (ABF3) of Theorem I.1, $\mathrm{H}(\theta)$ is the unique fixed point of ξ_Γ in the set

$$\mathrm{N} = \{\mathrm{H}(\theta) \in \Phi : d(\mathrm{H}(\theta), \Gamma(\theta)) < \infty\}.$$

Finally, by (ABF4) of the Theorem I.1, we settled

$$d(\Gamma, \mathrm{H}) \le \frac{L^{1-i}}{1-L} d(\xi_\Gamma, \Gamma)$$

$$\Rightarrow \quad d(\Gamma, \mathrm{H}) \le \frac{L^{1-i}}{1-L} \alpha(\theta)$$

$$\Rightarrow \quad \|\Gamma(\theta) - \mathrm{H}(\theta)\| \le \frac{L^{1-i}}{1-L} \alpha(\theta)$$

for all $\theta \in \psi_1$. This concludes the proof.

Corollary III.2 Let's suppose that ε is a positive number and ω is a real number with $\omega \ne 3$. Let $\Gamma : \psi_1 \to \psi_2$ be a transformation satisfying the functional disproportion

$$\|\partial\Gamma(\theta_1, \theta_2, \ldots, \theta_n)\| \le \begin{cases} \varepsilon; \\ \varepsilon \sum_{i=1}^{n} \|\theta_i\|^{\omega}; \omega \ne 3 \end{cases}$$

(27)

for all $\theta_i \in \psi_1, (i = 1,2,3,\ldots,n)$. Then there exists unique cubic transformation $\mathrm{H} : \psi_1 \to \psi_2$ so as to

$$\|\Gamma(\theta) - \mathrm{H}(\theta)\| \le \begin{cases} \dfrac{\varepsilon n^{3\beta}}{|n^{3\beta} - 1|}; \\ \dfrac{\varepsilon n^{\beta(3+\omega)} \|\theta\|^{\omega}}{|n^{\omega\beta} - n^{3\beta}|}; \omega \ne 3 \end{cases}$$

(28)

for all $\theta \in \psi_1$.

Proof. Let

$$\lambda(\theta_1, \theta_2, \ldots, \theta_n) = \begin{cases} \varepsilon; \\ \varepsilon \sum_{i=1}^{n} \|\theta_i\|^{\omega}; \end{cases}$$

(29)

for all $\theta_i \in \psi_1, (i = 1,2,3,\ldots,n)$ in Theorem III.1. Replacing

$(\theta_1, \theta_2, \ldots, \theta_n)$ by $(v_i^{s}\theta_1, v_i^{s}\theta_2, \ldots, v_i^{s}\theta_n)$

and dividing by v_i^{3s} in the equation (29), we reach

$$\frac{1}{v_i^{3s}} \lambda(v_i^{s}\theta_1, v_i^{s}\theta_2, \ldots, v_i^{s}\theta_n) = \begin{cases} \dfrac{\varepsilon}{v_i^{3s}}; \\ \dfrac{\varepsilon}{v_i^{3s}} \sum_{i=1}^{n} v_i^{\omega\beta} \|\theta_i\|^{\omega}; \end{cases} = \begin{cases} \to 0 \\ \to 0 \end{cases} \text{ as } s \to 0.$$

Therefore (15) holds for all $\theta_i \in \psi_1, (i = 1,2,3,\ldots,n)$. Now it follows from (16), we accomplish

$$\alpha(\theta) = \lambda(0,0,\ldots,n\theta) = \begin{cases} \varepsilon; \\ \varepsilon n^{\omega\beta} \|\theta\|^{\omega}; \end{cases}$$

and

$$\frac{1}{v_i^{3\beta}} \alpha(v_i\theta) = \begin{cases} \dfrac{1}{v_i^{3\beta}} \varepsilon; \\ \dfrac{1}{v_i^{3\beta}} v_i^{\omega\beta} \varepsilon n^{\omega\beta} \|\theta\|^{\omega}; \end{cases}$$

$$\Rightarrow \quad \frac{1}{v_i^{3\beta}} \alpha(v_i\theta) = \begin{cases} v_i^{-3\beta} \varepsilon; \\ v_i^{\beta(\omega-3)} \varepsilon n^{\omega\beta} \|\theta\|^{\omega}; \end{cases}$$

$$\Rightarrow \quad \frac{1}{v_i^{3\beta}} \alpha(v_i\theta) = \begin{cases} v_i^{-3} \alpha(\theta); \\ v_i^{\beta(\omega-3)} \alpha(\theta); \end{cases}$$

$$\Rightarrow \quad \frac{1}{{v_i}^{3\beta}}\alpha(v_i\theta) = \begin{cases} L\alpha(\theta); \\ L\alpha(\theta); \end{cases}$$

for each $\theta \in \psi_1$.

Case (i): For $i=0$, we acquire $L = {v_i}^{-3\beta} = \left(\frac{1}{n}\right)^{-3\beta} = n^{3\beta}$. From (17), we observe that

$$\|\mathrm{H}(\theta) - \Gamma(\theta)\| \le \frac{L^{1-i}}{1-L}\alpha(\theta) = \frac{L}{1-L}\alpha(\theta) = \frac{\varepsilon n^{3\beta}}{1-n^{3\beta}}$$

(30)

for every $\theta \in \psi_1$.

Case (ii): For $i=1$, we hold $L = {v_i}^{-3\beta} = (n)^{-3\beta} = \frac{1}{n^{3\beta}}$. From (17), we claim that

$$\|\mathrm{H}(\theta) - \Gamma(\theta)\| \le \frac{L^{1-i}}{1-L}\alpha(\theta) = \frac{1}{1-L}\alpha(\theta) = \frac{\varepsilon n^{3\beta}}{n^{3\beta}-1}$$

(31)

for all $\theta \in \psi_1$.

Case (iii): For $i=0$, we gain $L = {v_i}^{\beta(\omega-3)} = \left(\frac{1}{n}\right)^{\beta(\omega-3)} = \frac{n^{3\beta}}{n^{\omega\beta}}$. From (17), one can obtain

$$\|\mathrm{H}(\theta) - \Gamma(\theta)\| \le \frac{L^{1-i}}{1-L}\alpha(\theta) = \frac{L}{1-L}\alpha(\theta) = \frac{\varepsilon n^{\beta(\omega+3)}\|\theta\|^{\omega}}{n^{\omega\beta}-n^{3\beta}}$$

(32)

for all $\theta \in \psi_1$.

Case (iv): For $i=1$, we attain $L = {v_i}^{\beta(\omega-3)} = (n)^{\beta(\omega-3)} = \frac{n^{\omega\beta}}{n^{3\beta}}$. From (17), one can reach

$$\|\mathrm{H}(\theta) - \Gamma(\theta)\| \le \frac{L^{1-i}}{1-L}\alpha(\theta) = \frac{\alpha(\theta)}{1-L} = \frac{\varepsilon n^{\beta(\omega+3)}\|\theta\|^{\omega}}{n^{3\beta}-n^{\omega\beta}}$$

(33)

for each and every $\theta \in \psi_1$. This concludes the demonstration of the corollary.

IV. APPLICATION PROBLEM

In this section we provide some real-life application problems based on the cubic functional equation (1) for $n=3$.

For $n=3$, it is straightforward to confirm that the functional equation (1) transforms into

$$\Gamma\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{4}\right) = \frac{1}{64}\begin{Bmatrix} \Gamma(\theta_1)+\Gamma(\theta_2) \\ +\Gamma(\theta_3)+\Gamma(\theta_4) \end{Bmatrix} + \frac{1}{64}\begin{Bmatrix} \Gamma(\theta_1+\theta_2+\theta_3)+\Gamma(\theta_1+\theta_2+\theta_4) \\ +\Gamma(\theta_1+\theta_3+\theta_4)+\Gamma(\theta_2+\theta_3+\theta_4) \end{Bmatrix}$$
$$-\frac{1}{64}\begin{Bmatrix} \Gamma(\theta_1+\theta_2)+\Gamma(\theta_1+\theta_3)+\Gamma(\theta_1+\theta_4) \\ +\Gamma(\theta_2+\theta_3)+\Gamma(\theta_2+\theta_4)+\Gamma(\theta_3+\theta_4) \end{Bmatrix}$$

(34)

Application in Structural Engineering: Predicting Material Strength for Composite Beams

In structural engineering, composite beams are often used in construction to combine materials with different properties (like concrete, steel, and other composites) in order to optimize performance. One of the key factors in designing composite beams is predicting how the materials will interact under various loading conditions, specifically how they distribute stress across their sections.

Problem Setup:

Given a composite beam made of four different materials, the goal is to predict the flexural strength (or bending resistance) of the beam when subjected to stress. The materials have different stress levels, denoted as $\theta_1 = 4Mpa$, $\theta_2 = 5Mpa$, $\theta_3 = 6Mpa$, $\theta_4 = 7Mpa$. The behavior of the composite beam under these stress levels is modeled by the following cubic functional equation (34). The function $\Gamma(\theta) = \theta^3$ represents the material's stress response to a given load. The objective is to determine how the effective stress of the composite beam behaves when considering the stress contributions from each material.

Solution

1. **Calculate Individual stress values:**

   Using $\Gamma(\theta)=\theta^3$

$$\Gamma(4)=4^3=64\ ,\ \ \Gamma(5)=5^3=125\,,\Gamma(6)=6^3=216\,,\Gamma(7)=7^3=343$$

2. **Calculate pairwise sums of stress values:**

$$\Gamma(4+5)=\Gamma(9)=9^3=729\,,\Gamma(4+6)=\Gamma(10)=10^3=1000\,,\Gamma(4+7)=\Gamma(11)=11^3=1331$$

$$\Gamma(5+6)=\Gamma(11)=11^3=1331\,,\Gamma(5+7)=\Gamma(12)=12^3=1728\,,\Gamma(6+7)=\Gamma(13)=13^3=2197$$

3. **Calculate Triple-Sum of stress values:**

$$\Gamma(4+5+6)=\Gamma(15)=15^3=3375,\ \Gamma(4+5+7)=\Gamma(16)=16^3=4096$$

$$\Gamma(4+6+7)=\Gamma(17)=17^3=4193,\ C(5+6+7)=C(18)=18^3=5832$$

4. **Compute the Left-Hand Side (LHS) of the Equation:**

$$\Gamma\left(\frac{4+5+6+7}{4}\right)=\Gamma(5.5)=(5.5)^3=166.375$$

5. **Compute the Right-Hand Side (RHS):**

$$\frac{1}{64}\{64+125+216+343\}+\frac{1}{64}\{3375+4096+4913+5832\}-\frac{1}{64}\{729+1000+1331+1331+1728+2197\}$$
$$=\frac{1}{64}(748+18216-8316)\ \ =166.375$$

Thus, the effective stress (material strength) of the composite beam is approximately **166.375Mpa**

Conclusion:

This demonstrates how the cubic functional equation can be applied to predict the material strength of a composite beam made from four different materials. By using the stresses of individual materials and applying the functional equation (34), we can calculate the overall strength of the composite beam, which is crucial for design and analysis in structural engineering.

REFERENCES

[1] J. Aczel, J. Dhombres , Functional Equations in Several Variables, Cambridge University Press, 1989.

[2] T. Aoki, On the stability of the linear transformation Banach spaces, The Journal of the Mathematical Society of Japan. (1950) 2, 64–66.

[3] M. Arunkumar, S. Karthikeyan and S. Ramamoorthi , Generalized Ulam-Hyers Stability of N-Dimensional Cubic Functional Equation in FNS and RNS: Various Methods Middle-East Journal of Scientific Research 24 (S2): 386-404, 2016.

[4] M. Arunkumar, C. Devi Shyamala Mary, Solution and generalized Ulam -Hyers stability of a cubic functional equation, Proceedings of the ”Heber International Conference on Applications of mathematics and Statistics”,(2012), 46-55.

[5] M.Arunkumar, E.Sathya, V.Chandiran, V.Alexpandiyan and T.Velmurugan, Gavruta Rassias stabilities of a functional equation involving mean sum of functions of consecutive variables, Journal

of Emerging Technologies and Innovative Research (JETIR). November 2024, Volume 11, Issue 11,www.jetir.org, (ISSN-2349-5162).

[6] M.Arunkumar, V.Chandiran, E.Sathya Unraveling the Stability of a Cubic Functional Equation Involving Consecutive Variable Mean Sums, International Journal of Engineering Development and Research. Volume 13 | Issue 1 | March-2025

[7] S.Czerwik, Stability of Functional Equations of Ulam-Hyers-Rassias Type, 2003, Hadronic Press, Palm Harbor, Fla, USA.

[8] Z. Gajada, On the stability of additive mappings, Int. J. Math. Math. Sci. 14 (1991) 431–434.

[9] P.Gavruta, A generalization of the Hyers-Ulam-Rassias, Stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431- 436.

[10] D.H.Hyers, G.Isac and Th.M.Rassias., Stability of functional equations in several variables, Birkhauser, Basel, 1998.

[11] D.H.Hyers, On the stability of the linear functional equation, Proceedings of the National Academy of Sciences of the United States of America. (1941) 27, no. 1, 222–224.

[12] Y.S. Jung, I.S.Chang, The stability of a cubic type functional equation with the fixed point alternative, J. Math. Anal. Appl. 306 (2005) 752–760.

[13] K.Ravi and M.Arunkumar, On The Cubic Functional Equation, Applied Science Periodical, Vol.VIII, No.4, (2006), 281- 285.

[14] K.Ravi and M.Arunkumar, On The Generalized Hyers-Ulam- Rassias Stability of a Cubic Functional Equation, International Review of Pure and Applied Mathematics, Vol.2, No.1, (2006), 13-23.

[15] K.-W. Jun, H.-M. Kim, The generalized Hyers–Ulam–Rassias stability of a cubic functional equation, J. Math. Anal. Appl. 274 (2002) 867–878.

[16] R. Kadison, G. Pedersen, Means and covex combinations of unitary operators, Math. Scand. 57 (1985) 249–266.

[17] B. Margolis, J.B. Diaz, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 126 (1968) 305–309.

[18] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.

[19] Th.M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000) 264–284

[20] Th.M. Rassias, P. Šemrl, On the Hyers–Ulam stability of linear mappings, J. Math. Anal. Appl. 173 (1993) 325–338.

[21] I.A. Rus, Principles and Applications of Fixed Point Theory, Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).

[22] F. Skof, Proprietà locali e approssimazione di operatori, Rend. Semin. Mat. Fis. Milano 53 (1983) 113–129.

[23] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1960.

[24] T.Z. Xu, J.M. Rassias, M.J. Rassias, W.X. Xu, A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces, J. Inequal. Appl. 2010, Art. ID 423231, 23 pp

[25] T.Z. Xu, J.M. Rassias, Approximate Septic and Octic mappings in quasi-β-normed spaces, J. Computational Analysis and Applications, Vol.15, No. 6, 1110 - 1119, 2013, copyright 2013 Eudoxus Press, LLC.