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CONCEPTS OF PRE A* - ALGEBRAS

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Absract:

This paper deals with the concept of *Boolean algebras and Pre A*-algebras* .This paper commences with the concept of Boolean algebra and some basic fundamental results of Boolean algebra. It also includes the useful properties of Boolean algebra. It introduces the concept of Pre A*-algebra and obtain the useful characterizations and obtain the various methods of generation of Pre A*-algebras from Boolean algebra.

KEYWORDS AND PHRASES: Boolean algebras, Pre A* - Algebra, Huntington's Theorem **,Pre A* - isomorphism**

§ Introduction:

Boolean algebras, essentially introduced by Boole in 1850's to codify the laws of thought, have been a popular topic of research since then. A major breakthrough was the duality of Boolean algebras and Boolean spaces as discovered by Stone in 1930's. Stone also proved that Boolean algebras and Boolean rings are essentially the same in the sense that one can convert via terms from one to the other. Since every Boolean algebra can be represented as a field of sets, the class of Boolean algebras is sometimes regarded as being rather uncomplicated. However, when one starts to look at basic questions concerning decidability, rigidity, direct products etc., they are associated with some of the most challenging results.

In a draft paper [3], *The Equational theory of Disjoint Alternatives*, around 1989, E.G.Manes introduced the concept of Ada (Algebra of disjoint alternatives) $(A, \wedge, V, (-)^I, (-)_\pi, 0, 1, 2)$(Where $\wedge$,V are binary operations on A, $(-)^I$, $(-)_\pi$ are unary operations and 0,1,2 are distinguished elements on A)which is however differ from the definition of the Ada of his later paper[4] *Adas and the equational theory of if-then-else* in 1993. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras $(A, \wedge, V, (-)^{\sim})$)(where $\wedge$,V are binary operations on A,$(-)^{\sim}$ is a unary operation) introduced by Fernando Guzman and Craig C. Squir[2]. In 1994, P.Koteswara Rao[2] first introduced the concept of A*-algebra $(A, \wedge, V, *, (-)^{\sim}, (-)_\pi, 0, 1, 2)$)(where $\wedge$,V,$*$ are binary operations on A, $(-)^{\sim}$,$(-)_\pi$ are unary operations and 0,1,2 are distinguished elements on A) not only studied the equivalence with Ada, C-algebra, Ada's connection with 3-Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. In 2000, J.Venkateswara Rao[5] introduced the concept Pre A*-algebra $(A, \vee, \wedge, (-)^{\sim})$(where $\wedge$,V are binary operations on A,$(-)^{\sim}$ is a unary operation on A)analogous to C-algebra as a reduct of A*- algebra, studied their subdirect representations, obtained the results that **2**= {0, 1} and **3**= {0, 1, 2} are the subdirectly irreducible Pre-A*-algebras and every Pre-A*-algebra can be imbedded in $\mathbf{3}^X$ **(** where $\mathbf{3}^X$ is the set of all mappings from a nonempty set X into **3**= {0, 1, 2}**).**

Boolean algebra depends on two-element logic. C-algebra, Ada, A*- algebra and our Pre A*-algebra are regular extensions of Boolean logic to 3 truth-values, where the third truth-value stands for an undefined truth-value. The Pre A*- algebra structure is denoted by $(A, \vee, \wedge, (-)^{\sim})$where A is non-empty set, $\wedge$, $\vee$ are binary operations and $^{\sim}$ is a unary operation.

§ Preliminaries:

1.1. BOOLEAN ALGEBRA:

1.1.1Definition: A Boolean algebra is an algebra $(B, \vee, \wedge, (-)^{\sim}, 0, 1)$ with two binary operations, one unary operation (called complementation), and two nullary operations which satisfies :

(1) $(B, \vee, \wedge)$ is a distributive lattice;

(2) $x \wedge 0 = 0$, $x \vee 1 = 1$ for all $x \in B$; (3) $x \wedge x' = 0$ $x \vee x' = 1$ for all $x \in B$.

We can easily prove that $x'' = x$, $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$ for all $x, y \in B$.

1.1.2 Definition: Let X be a set. The Boolean Algebra of subsets of X, P(X), has as its universe P(X) and as operations $\vee, \wedge, ', \phi$, X. The Boolean Algebra **2 =** {0, 1} is given by **(2,** $\vee,\wedge,'$,0,1), where $(2, \vee,\wedge)$ is a 2 element lattice with $0 < 1$ and where $0' = 1$, $1' = 0$.

Alternative systems of postulates for Boolean Algebras were intensively studied during the decades 1900 - 1940. E.V. Huntington wrote an influential early paper [3] on this subject. No attempt will be made here to survey the extensive literature on such postulate systems. We present here Huntington's postulates in 1.1.3

1.1.3. Huntington's Theorem [1] : Let B have one binary operation $\vee$ and one unary operation $(-)'$ and define (i) $a \wedge b = (a' \vee b')'$ for all $a, b \in B$. Suppose for all $a, b, c \in B$, (ii) $a \vee b = b \vee a$; (iii) $a \vee (b \vee c) = (a \vee b) \vee c$ and (iv) $(a \wedge b) \vee (a \wedge b') = a$. Then B is a Boolean algebra.

1.1.4. Theorem [7] : Let B have one binary operation $\wedge$ and one unary operation $(-)'$ and define (i) $a \vee b = (a' \wedge b')'$ for all $a, b \in B$. Suppose for $a, b, c \in B$,

(ii) $a \vee b = b \vee a$, (iii) $(a \vee b) \vee c = a \vee (b \vee c)$ (iv) $(a \wedge b) \vee (a \wedge b') = a$.

Then B is a Boolean algebra.

1.2. Pre A* - Algebra:

1.2.1 Definition: An algebra $(A, \vee, \wedge, (-)^{\sim})$ satisfying:

(a) $(x^{\sim})^{\sim} = x, \forall x \in A$

(b) $x \wedge x = x, \forall x \in A$

(c) $x \wedge y = y \wedge x, \forall x,y \in A$

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall x,y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x,y,z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x,y,z \in A$

(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x,y \in A$

is called a Pre A* - algebra.

1.2.2 Note : If (mn) is an axiom in Pre A* - algebra, then $(mn)^{\sim}$ is its dual.

1.2.3 Examples : **3** = {0,1,2} with $\vee, \wedge, (-)^{\sim}$ defined below is a Pre A* - algebra

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

1.2.4 Example : 2 = {0,1} with $\vee, \wedge, (-)^{\sim}$ defined below is a Pre A* - algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

Actually $(B, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* - algebra.

1.2.5 Note :- The elements 0,1,2 in examples satisfy the following laws.

(a) $2^{\sim} = 2$; (b) $1 \wedge x = x, \forall x \in \mathbf{3}$ ('1' the identity for $\wedge$)

(c) $1^{\sim} = 0$; (d) $2 \wedge x = 2, \forall x \in \mathbf{3}$

(e) $0 \vee x = x, \forall x \in \mathbf{3}$ ('0' is the identity for $\vee$)

1.2.6 Note :- Let A be a Pre A* - algebra with 0 and1. Then by 1.3,

$B(A) = \{x \in A / 1 \vee x = 1\}$ becomes a Boolean Algebra with $\vee, \wedge, (-)^{\sim}$, 0.

1.2.7 Theorem [8]: Let $(B, \wedge, (-)', 0)$ be a Boolean Algebra.

Then, $A(B) = \{(a_1, a_2) / a_1, a_2 \in B \text{ and } a_1 \wedge a_2 = 0\}$ becomes a Pre A* - algebra with $1 = (1,0)$, $0 = (0,1)$ and for all $a, b \in A(B)$,

(i) $a \wedge b = (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$ (where juxta position, +, $(-)'$ respectively

$\wedge, \vee$, $(-)'$ in Boolean Algebra B).

(ii) $a \vee b = (a_1b_1 + a_1b_2 + a_2b_1 , a_2b_2)$

(iii) $a^{\sim} = (a_2, a_1)$

Proof :

(i) for $a \in B$, $a^{\sim} = (a_2, a_1)$

$\Rightarrow (a^{\sim})^{\sim} = (a_1, a_2)$

$= a$

Therefore $(a^{\sim})^{\sim} = a$

(ii) $a \wedge a = (a_1, a_2) \wedge (a_1, a_2)$

$= (a_1, a_1a_2 + a_2 a_1 + a_2)$

$= (a_1, 0 + a_2)$ (Since $a = (a_1, a_2) \in B \times B. \Rightarrow a_1 a_2 = 0$)

$= (a_1, a_2)$

$= a$

Therefore $a \wedge a = a, \forall a \in B$

(iii) For $a,b, \in B$

$a \wedge b = (a_1 b_1, a_1 b_2 + a_2 b_1 + a_2 b_2)$

$= (b_1 a_1, b_2 a_1 + b_1 a_2 + b_2 a_2) = b \wedge a$

Therefore $a \wedge b = b \wedge a$

(iv) To show that $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$

$(a \wedge b)^{\sim} = (a_1 b_1, a_1 b_2 + a_2 b_1 + a_2 b_2)^{\sim}$

$= (a_1 b_2 + a_2 b_1 + a_2 b_2, a_1 b_1)$

Now $a^{\sim} \vee b^{\sim} = (a_2, a_1) \vee (b_2, b_1)$

$= (a_2 b_2 + a_2 b_1 + a_1 b_2, a_1 b_1)$

$= (a \wedge b)^{\sim}$

Therefore $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$

(v) To show that $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \vee a, b, c, \in B$

$a \wedge (b \wedge c) = (a_1, a_2) \wedge \{(b_1, b_2) \wedge (c_1, c_2)\}$

$= (a_1, a_2) \wedge \{(b_1 c_1, b_1 c_2 + b_2 c_1 + b_2 c_2)\}$

$= (a_1 b_1 c_1 , a_1 b_1 c_2 + a_1 b_2 c_1 + a_1 b_2 c_2 + a_2 b_1 c_1$

$+ a_2 b_1 c_2 + a_2 b_2 c_1 + a_2 b_2 c_2)$

Now $(a \wedge b) \wedge c \quad = \{(a_1, a_2) \wedge (b_1, b_2)\} \wedge (c_1, c_2)$

$= \{(a_1 b_1 , a_1 b_2 + a_2 b_1 + a_2 b_2)\} \wedge (c_1, c_2)$

$= (a_1b_1c_1 , a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 + a_2b_2c_1 + a_1b_2c_2$

$+ a_2b_1c_2 + a_2b_2c_2)$

Therefore $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(vi) To show that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Consider

$a \wedge (b \vee c) \quad = (a_1,a_2) \wedge (b_1c_1 + b_1c_2 + b_2c_1 , b_2c_2)$

$= (a_1b_1c_1 + a_1b_1c_2 + a_1b_2c_1, a_1b_2c_2 + a_2b_1c_1 + a_2b_1c_2$

$+ a_2b_2c_1 + a_2b_2c_2)$

Now

$(a \wedge b) \vee (a \wedge c) \quad = (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2) \vee$

$(a_1c_1, a_1c_2 + a_2c_1 + a_2c_2)$

$= (a_1b_1c_1 + a_1b_1c_2 + a_1b_2c_1, a_1b_2c_2 + a_2b_1c_1 + a_2b_1c_2$

$+ a_2b_2c_1 + a_2b_2c_2)$ (Since $a_1a_2 = 0$)

Therefore $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(viii) To verify that $a \wedge b = a \wedge (a^{\sim} \vee b)$:-

Consider

$a \wedge (a^{\sim} \vee b) \quad = (a_1,a_2) \wedge [(a_2,a_1) \vee (b_1,b_2)]$

$= (a_1b_1\ ,\ a_1b_2 + a_2b_1 + a_2b_2)$ (Since $a_1a_2 = 0$)

$= a \wedge b$

Hence A(B) becomes a Pre A* - algebra.

1.2.8 Note : Let (A, ∨, ∧, (-)˜, 0, 1) be a Pre A* - algebra Then from 1.2.6 & 1.2.7 we have A ≅ A (B(A)).

1.2.9 Theorem[8]: Suppose (B, ∨, ∧, (-)′, 0, 1) is a Boolean Algebra.

Define ≈ on B x B as (a, b) ≈ (c,d) if and only if a = c and a′ b = c′ d. Then,

(i) ≈ is an equivalence relation on B x B ; [a,b] = {(c,d) ∈ B x B / (a,b) ≈ (c,d)}, the equivalence class containing (a,b).

Let A_B = B x B / ≈ = {[a,b] / (a,b) ∈ B x B}.

(ii) For every [a,b] ∈ A_B there exists e, f ∈ B and ef = 0

such that (e, f) ∈ [a,b] and (e, f) is unique.

(iii) Define ∨, ∧, (-)˜ on A_B as follows.

Assume that A_B = {[a,b] / a,b ∈ B, ab = 0}

(a) [a,b] ∧ [c,d] = [ac, ad+bc+bd]

(Where juxte position, + respectively ∧, ∨ in the Boolean Algebra B)

(b) [a,b] ∨[c,d] = [ac+ad+bc, bd]

(c) [a, b]˜ = [b, a]

Then (A_B, ∨, ∧, (-)˜) is a Pre A* - algebra.

Proof : Let $(a,b) \in B \times B$.

Since $a = a$, $a'b = a'b$ we have $(a,b) \approx (a,b)$

Therefore $\approx$ is reflexive.

If $(a,b)\approx(c,d)$ then $a=c$ and $a'b=c'd$

$\Rightarrow c=a$ and $c'd=a'b$

$\Rightarrow (c,d)\approx(a,b)$

Therefore $\approx$ is symmetric.

Let $(a,b)\approx(c,d)$ and $(c,d)\approx(e,f)$, then $a=c$, $a'b=c'd$ and $c=e$, $c'd=e'f$

$\Rightarrow a=e$ and $a'b=e'f$

$\Rightarrow (a,b)\approx(e,f)$

Therefore $\approx$ is transitive.

Hence $\approx$ is a equivalence relation on $B\times B$.

Let $[a,b]\in A_B$. Let $e=a$, $f=a'b$.

Clearly $e, f\in B$ and $ef=a(a'b)=0$.

And also $e'f=a'(a'b)=a'b$.

Therefore $a=e,\ e'f=a'b$. Hence $(e,f)\in[a,b]$

Suppose $e_1, f_1\in B$, $e_1 f_1=0$ and $(e_1,f_1)\in[a,b]$

Since $(e,f)\in[a,b]$ and $(e_1,f_1)\in[a,b]$,

$(a,b)\approx(e,f)$ and $(a,b)\approx(e_1,f_1)$.

By transitivity $(e,f)\approx(e_1,f_1)$.

$\Rightarrow e=e_1$ and $e'f=e_1' f_1$ □

$\Rightarrow e=e_1$ and $f=f_1$

Therefore (e,f) is unique.

Let $[a,b] \in A_B$

(i) $[a,b]^{\sim\sim} = [b,a]^{\sim} = [a,b]$

(ii) $[a,b] \wedge [a,b] = [aa, ab+ba+bb]$

$= [a, 0+0+b]$

$= [a,b]$

Therefore $[a,b] \wedge [a,b] = [a,b]$

(iii) Let $[a,b],\ [c,d] \in A_B$

$[a,b] \wedge [c,d] = [ac, ad+bc+bd]$

$= [ca, da+cb+db]$

$= [c,d] \wedge [a,b]$

Therefore $[a,b] \wedge [c,d] = [c,d] \wedge [a,b]$

(iv)To prove $\{[a,b] \wedge [c,d]\}^{\sim} = [a,b]^{\sim} \vee [c,d]^{\sim}$

Consider $\{[a,b] \wedge [c,d]\}^{\sim} = \{[ac, ad+bc+bd]\}^{\sim} = [ad+bc+bd, ac]$

Now $[a,b]^{\sim} \vee [c,d]^{\sim} = [b,a] \vee [d,c] = [ad+bc+bd, ac]$

Hence $\{[a,b] \wedge [c,d]\}^{\sim} = [a,b]^{\sim} \vee [c,d]^{\sim}$

(v) To Prove $[a,b] \wedge \{[c,d] \wedge [e,f]\} = \{[a,b] \wedge [c,d]\} \wedge [e,f]$

Consider $[a,b] \wedge \{[c,d] \wedge [e,f]\} = [a,b] \wedge [ce, cf+de+df]$

$= [ace, acf+ade+adf+bce+bcf+bde+bdf]$

Now $\{[a,b] \wedge [c,d]\} \wedge [e,f] = [ac, ad+bc+bd] \wedge [e,f]$

$$=[ace, acf+ade+adf+bce+bcf+bde+bdf]$$

Hence $[a,b]\wedge\{[c,d]\wedge[e,f]\}=\{[a,b]\wedge[c,d]\}\wedge[e,f]$

(vi) To prove $[a,b]\wedge\{[c,d]\vee[e,f]\}=\{[a,b]\wedge[c,d]\}\vee\{[a,b]\wedge[e,f]$

Consider $[a,b]\wedge\{[c,d]\vee[e,f]\}=[a,b]\wedge[ce+cf+de, df]$

$$=[ace+acf+ade, adf+bce+dcf+bde+bdf]$$

Now $\{[a,b]\wedge[c,d]\}\vee\{[a,b]\wedge[e,f]$

$$=[ac, ad+bc+bd]\vee[ae, af+be+bf]$$

$$=[ace+acf+ade, adf+bce+dcf+bde+bdf]$$

Hence $[a,b]\wedge\{[c,d]\vee[e,f]\}=\{[a,b]\wedge[c,d]\}\vee\{[a,b]\wedge[e,f]$

(vii) To prove $[a,b]\wedge[c,d]=[a,b]\wedge\{[a,b]^{\sim}\vee[c,d]\}$

Consider $[a,b]\wedge\{[a,b]^{\sim}\vee[c,d]\}=[a,b]\wedge\{[b,a]\vee[c,d]\}$

$$=[a,b]\wedge[bc+bd+ac, ad]$$

$$=[ac, ad+bc+bd]$$

$$=[a,b]\wedge[c,d]$$

Hence $[a,b]\wedge[c,d]=[a,b]\wedge\{[a,b]^{\sim}\vee[c,d]\}$

Therefore A_B is a Pre A*-algebra which is generated by Boolean algebra.

1.2.10 Corollary [8]:- Let (A, $\vee$, $\wedge$, $(-)^{\sim}$) be a Pre A* - algebra. B(A) is defined as in **1.**2.6 Then, A $\cong$ $A_{B(A)}$.

Proof : A_B is a Pre A*-algebra which is generated by Boolean algebra.

One can see that for every Boolean algebra B, $A_B \cong A(B)$.

Define, $f: A_B \to A(B)$ by $f([a,b]) = (a,b)$ for all $[a,b] \in A_B$.

Then f is clearly well defined, bijection and also

(i) $f([a,b] \vee [c,d]) = f([ac+ad+bc, bd])$

$$= (ac+ad+bc, bd)$$

$$= (a,b) \vee (c,d)$$

$$= f([a,b]) \vee f([c,d])$$

(ii) $f([a,b] \wedge [c,d]) = f([ac, ad+bc+bd])$

$$= [ac, ad+bc+bd]$$

$$= (a,b) \wedge (c,d)$$

$$= f([a,b]) \wedge f([c,d])$$

(iii) $f([a,b]^{\sim}) = f([b,a]) = (b,a) = (a,b)^{\sim} = (f([a,b]))^{\sim}$

Hence it is a Pre A*-isomorphism. Therefore $A_B \cong A(B)$

Conclusion: We commences with the concept of Boolean algebra and some basic fundamental results of Boolean algebra. We studied the useful properties of Boolean algebra. We introduced the concept of Pre A*-algebra and obtain the useful characterizations. We obtained the various methods of generation of Pre A*-algebras from Boolean algebra. The main content of this paper was embodied in “Boolean algebras and Pre A*-algebras” [8].and was published in the journal Acta Ciencia Indica (Mathematics), (ISSN: 0970-0455), 32: pp71-76.)

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