IJCRT.ORG ISSN : 2320-2882
INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

Solution Of Partial Differential Equations Using Laplace And Fourier Transforms

R. T. Katre

Department of Mathematics, Bajaj College of Science, Wardha, India

Abstract

In this paper, I present the definition and important properties of Laplace and Fourier transforms which are applicable for solving the partial differential equations. The result shows that the Laplace and Fourier transforms are more effective and useful to solve the initial and boundary value problems.

Keywords: Laplace Transform, Fourier Transform, Partial differential equations.

1. Introduction

The integral transforms are useful for solving the ordinary differential equations, partial differential equations, integral equations, Integro-differential equations etc. In this paper we apply Laplace and Fourier transforms to solve the partial differential equations with initial and boundary conditions.[1-5] Let $u(x,t)$ be the function of two independent variables x and t .The partial differential equations involves the partial derivatives u_x, u_t, u_{xx}, u_{tt}, u_{xt} etc. of function $u(x,t)$. While applying these transforms to partial differential equations it is necessary to know the transforms of partial derivatives of function. The advantage of the integral transforms is to obtain the solution of initial value problems without finding the general solution and values of arbitrary constants. Here I mentioned the definition, important properties, theorems on Laplace and Fourier transforms which are applicable for solving the partial differential equations.

2. Laplace Transforms

The Laplace transform methods provide an easy and effective means for the solution of many problems arising in engineering and science.

Definition. Let $f(t)$ be a function defined for all $t > 0$. The Laplace transform of $f(t)$ is $\int_0^\infty e^{-st} f(t)\, dt$, [1-6, 10, 12, 13]. It is denoted by $L\{f(t)\}$ or $F(s)$

Thus $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$, provided that the integral exists.

Here L is called Laplace transformation operator. The parameter s is real or complex.

Theorem. Let $u(x,t)$ be a function defined for $t > 0$ and $x \in [a,b]$. Then

$(i)\ L\left(\frac{\partial u}{\partial t}\right) = s\,U - u(x,0)$, where $U = U(x,s) = L\{u(x,t)$

$(ii)\ L\left(\frac{\partial u}{\partial x}\right) = \frac{d\,U}{dx}$

$(iii)\ L\left(\frac{\partial^2 u}{\partial t^2}\right) = s^2 U - s\,u(x,0) - u_t(x,0),$ where $u_t(x,0) = \frac{\partial u}{\partial t}$

$(iv)\ L\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{d^2\,U}{dx^2}$. [1-3, 5-6].

Proof.

$$(i)\ L\left(\frac{\partial u}{\partial t}\right) = \int_0^\infty e^{-st}\,\frac{\partial u}{\partial t}\,dt = \left[e^{-st}\;u(x,t)\right]_0^\infty - \int_0^\infty -s\,e^{-st}\;u(x,t)\;dt$$

$$= [\,0 - u(x,0)] + s\int_0^\infty e^{-st}\;u(x,t)\;dt = -\,u(x,0) + s\,L\{u(x,t)\}$$

$$= s\,U - u(x,0)$$

$$(ii)\ L\left(\frac{\partial u}{\partial x}\right) = \int_0^\infty e^{-st}\,\frac{\partial u}{\partial x}\,dt = \frac{d}{dx}\int_0^\infty e^{-st}\;u(x,t)\,dt = \frac{d}{dx}\,L\{u(x,t)\} = \frac{d\,U}{dx}.$$

(iii) Let $v = \frac{\partial u}{\partial t}$, then

$$L\left(\frac{\partial^2 u}{\partial t^2}\right) = L\left\{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\right\} = L\left(\frac{\partial v}{\partial t}\right) = s\,L\{\,v(x,t)\,\} - v(x,0)$$

$$= s\,L\left\{\frac{\partial u}{\partial t}\right\} - v(x,0) = s\,[\,s\,L\{u(x,t)\} - u\,(x,0)] - v(x,0)$$

$$= s^2\,U - s\,u(x,0) - \frac{\partial}{\partial t}u(x,0) = s^2\,U - s\,u(x,0) - u_t(x,0)\,.$$

(iv)Let $v = \frac{\partial u}{\partial x}$, then

$$L\left(\frac{\partial^2 u}{\partial x^2}\right) = L\left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right\} = L\left(\frac{\partial v}{\partial x}\right) = \frac{d}{dx}\,L\{\,v\,\}$$

$$= \frac{d}{dx}\,L\left\{\frac{\partial u}{\partial x}\right\} = \frac{d}{dx}\left(\frac{d}{dx}L\{u\}\right) = \frac{d^2}{dx^2}L\{u\} = \frac{d^2 U}{dx^2}\,.$$

3. Fourier Transforms

Definition. The Fourier transform of function $f(x)$ is denoted by $\bar{f}(s)$ or $F(s)$ or $F\{f(x)\}$ and is defined as [1-5, 9, 11]

$$F\{f(x)\} = \bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\;f(x)\,dx$$

where, s is real and the integral on right hand side is converges.

The inverse Fourier transform is given by

$$f(x) = F^{-1}\{\,\bar{f}(s)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\;\bar{f}(s)\,ds\,.$$

Fourier sine transform: Fourier sine transform of function $f(x)$ is denoted by $\bar{f}_s(s)$ or $F_s(s)$ or $F_s\{f(x)\}$ and is defined as $F_s\{f(x)\} = \bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\sin sx\,dx$

The inverse formula for Fourier sine transform is given by

$$f(x) = F_s^{-1}\{\,\bar{f}_s(s\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(s\,)\sin sx\,ds\,.$$

Fourier cosine transform: Fourier cosine transform of function $f(x)$ is denoted by $\bar{f}_c(s)$ or $F_c(s)$ or $F_c\{f(x)\}$ and is defined as

$$F_c\{f(x)\} = \bar{f}_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\;\cos\;sx\,dx$$

The inverse formula for Fourier cosine transform is given by

$$f(x) = F_c^{-1}\{\,\bar{f}_c(s\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_c(s\,)\cos sx\,ds\,.$$

Fourier Transforms of Derivative of a Function: [1-3, 7-9].

$(i)\ F\{f'(x)\} = (-is)\,\bar{f}(\,s\,)$ $\qquad$ $(ii)\ F\left\{f^{(n)}(x)\right\} = (-is)^n\,\bar{f}(\,s\,)$

$$(iii)\ F_c\{f'(x)\} = \ s\,\bar{f}_s(s\,) - \sqrt{\frac{2}{\pi}}\ f(0) \quad (iv)\ F_s\{f'(x)\}\ \ = -s\,\bar{f}_c(s\,)$$

$$(v)\ F_c\{f''(x)\} = -\,s^2\ \bar{f}_c(s\,) - \sqrt{\frac{2}{\pi}}\,f'(0)\ \ (vi)\ F_s\{f''(x)\} = -\,s^2\ \bar{f}_s(s\,)\sqrt{\frac{2}{\pi}}\ s\,f(0)$$

$$(vii)\ \ F\left\{\frac{\partial^2 u}{\partial x^2}\right\} \ = \ -\,s^2 F\{u\} \ = \ -\,s^2\,\bar{u}(s)$$

$$(viii)\ \ F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} \ = \ -\,s^2\,F_s(u)\ + s\ (u)_{x=0} \ \ = \ -\,s^2\,\bar{u}_s(s\,) + \ s\,u(0)$$

$$(ix)\ \ F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} \ = \ -\,s^2\,F_c(u) - \left(\ \frac{\partial u}{\partial x}\right)_{x=0} \ = \ -\,s^2\,\bar{u}_c(s\,) - \left(\ \frac{\partial u}{\partial x}\right)_{x=0}$$

4. Illustrative Examples

Example1. Solve $\frac{\partial u}{\partial t} = 3\ \frac{\partial^2 u}{\partial x^2}$, given that $u(0,t) = 0,\ u(5,t) = 0, u(x,0) = \sin \pi x$.

Taking Laplace transform to both sides of given partial differential equation and applying conditions, we get $s\,L\{\,u\,\} -\ u(x,0) = 3\ \frac{d^2}{dx^2} L\{u\}$

$$\Longrightarrow \quad s\,U - \ \sin \pi x \ = \ 3\ \frac{d^2 U}{dx^2}$$

$$\Longrightarrow \quad 3\ \frac{d^2 U}{dx^2} - \ sU \ = \ -\sin \pi x$$

$\Longrightarrow \quad \frac{d^2 U}{dx^2} - \ \frac{s}{3} U \ = \ -\ \frac{1}{3}\ \ \sin \pi x$. This is 2nd order linear differential equation.

Its solution is $\ U \ = \ A\,e^{\sqrt{s/3}\ x} + \ B\,e^{-\sqrt{s/3}\ x} + \frac{1}{s\ + 3\,\pi^2}\ \sin \pi x$ … (1)

Taking Laplace transform to the conditions which involve t,

$$L\{\,u(0,t)\,\} \ = \ U\,(\,0,\ s) = 0$$

$$L\{\,u(5,t)\,\} \ = \ U\,(\,5,\ s) = 0$$

Using these conditions in equation (1), we obtain $\ 0 \ = \ A + B$ … (i)

and $\quad 0 \ = \ A\,e^{5\sqrt{s/3}} \ + \ B\,e^{-5\sqrt{s/3}\ x} + \frac{1}{s\ + 3\,\pi^2}\ \sin 5\pi$

$\Longrightarrow \quad 0 \ = \ A\,e^{5\sqrt{s/3}} \ + \ B\,e^{-5\sqrt{s/3}\ x}$ \ … (ii)

Solve (i) & (ii) we get, $\ A = 0\ ,\ \ B \ = 0$

Then equation (1) becomes, $\ U = \frac{1}{s\ + 3\,\pi^2}\ \sin \pi x$

Taking inverse Laplace transform, we get

$$L^{-1}\{\,U\,\} = \sin \pi x \ \ L^{-1}\left\{\frac{1}{s\ + 3\,\pi^2}\right\} \ \Longrightarrow \ \ u(x,t) \ = \ \sin \pi x \ \ e^{-3\,\pi^2 t}\ .$$

Example 2. Solve $\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}$, given that $u(0,\ t) = u(2,\ t) = 0,\ \ \ u_t(x,\ 0) = 0$,
$u(x,\ 0) = 10 \sin 2\pi x - 20 \sin 5\pi x$.

Taking Laplace transform to both sides of given partial differential equation and applying conditions, we get $s^2\,L\{\,u\,\} - s\,u(x,0)\ -\ u_t(x,\ 0) = 9\ \frac{d^2}{dx^2} L\{u\}$

$$\Longrightarrow \qquad s^2\ U - s\ (\ 10 \sin 2\pi x - 20 \sin 5\pi x\ ) = \ 9\ \ \frac{d^2 U}{dx^2}$$

$$\Longrightarrow \qquad \frac{d^2 U}{dx^2} - \ \frac{s^2}{9}\ U \ = \ -\ \frac{s}{9}\ (\ 10 \sin 2\pi x - 20 \sin 5\pi x\ )$$

This is 2nd order linear differential equation. Its solution is

$$U \ = A\,e^{\frac{s}{3}x} \ + \ B\,e^{-\frac{s}{3}x} + \ \frac{10\ s}{s^2 + 36\ \pi^2} \sin 2\pi x \ - \ \frac{20\ s}{s^2 + 225\ \pi^2} \sin 5\pi x \qquad \ldots (1)$$

Taking Laplace transform to the conditions which involve t,

$L\{\,u(0,\ t)\,\} = U\,(\,0,\ s) = \ 0, \quad L\{\,u(2,\ t)\,\} = U\,(\,2,\ s) = \ 0$

Using these conditions in equation (1), we get

$0 \;=\; A + B$ and $0 = A\,e^{\frac{2s}{3}} + B\,e^{-\frac{2s}{3}x} \;\;\Rightarrow\;\; A = 0\,, \;\; B = 0$

Then equation (1) becomes, $U = \frac{10\,s}{s^2+36\,\pi^2}\sin 2\pi x \;-\; \frac{20\,s}{s^2+225\,\pi^2}\sin 5\pi x$

Taking inverse Laplace transform, we get

$$L^{-1}\{\,U\,\} \;=\; 10\;\sin 2\pi x\;\; L^{-1}\left\{\frac{s}{s^2+36\,\pi^2}\right\} - 20\;\sin 5\pi x\;\; L^{-1}\left\{\frac{s}{s^2+225\,\pi^2}\right\}$$

$$\Rightarrow \qquad u(x,t) \;=\; 10\;\sin 2\pi x\;\;\cos 6\pi t\; - 20\;\sin 5\pi x\;\;\cos 15\pi t\,.$$

Example 3. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, subject to the conditions

$u = 0,$ for $x = 0,\;\; t > 0\,;\;\; u = \begin{cases} 1\,, & 0 < x < 1 \\ 0\,, & x \ge 1 \end{cases}$ for $t = 0$ and $u(x,t)$ is bounded

Taking Fourier sine transform to given partial differential equation, we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin sx\,dx \;=\; -\,s^2\,\bar{u}_s(s\,) + \;s\,u(0)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\sqrt{\frac{2}{\pi}}\int_0^\infty u\;\sin sx\,dx \;=\; -\,s^2\,\bar{u}_s(s\,) \qquad (\because u = 0 \;\;\text{when}\;\; x = 0\,)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}F_s(u) \;=\; -\,s^2\,\bar{u}_s(s\,) \qquad \Rightarrow \qquad \frac{\partial}{\partial t}\,\bar{u}_s(s\,) \;=\; -\,s^2\,\bar{u}_s(s\,)$$

or $\frac{d\,\bar{u}_s}{dt} + \;s^2\;\bar{u}_s \;= 0.$ Its solution is,

$$\bar{u}_s \;=\; \bar{u}_s(s,t\,) \;=\; A\;\,e^{-s^2 t}\,, \qquad \ldots (1)$$

At $t = \;0\,, \qquad \bar{u}_s \;=\; A$ … (2)

Now $\bar{u}_s \;=\; \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\;\sin sx\,dx$

At $t = \;0\,, \quad \bar{u}_s \;=\; \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\;\sin sx\,dx \;=\; \sqrt{\frac{2}{\pi}}\int_0^1 1\;\sin sx\,dx$

$$\Rightarrow \quad \bar{u}_s \;=\; \sqrt{\frac{2}{\pi}}\left[\frac{-\cos sx}{s}\right]_0^1 \;=\; \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right) \qquad \ldots\;(3)$$

Equations (2) & (3) give, $A = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right),$ put this value in equation (1),

we get $\bar{u}_s \;=\; \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)\;e^{-s^2 t}$

Taking inverse Fourier sine transform, we obtain

$$u(x,t) \;=\; \sqrt{\frac{2}{\pi}}\int_0^\infty\left[\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s}\right)\;e^{-s^2 t}\right]\sin sx\,ds$$

$$u(x,t) \;=\; \frac{2}{\pi}\int_0^\infty\left(\frac{1-\cos s}{s}\right)\;e^{-s^2 t}\;\sin sx\;\,ds.$$

Example 4. Solve the equation $\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + \;tU = 0,\;\; x > 0\;\;,\; t > 0$ given that

$U(x,0) = e^{-x}\;,\;\; U_x(\,0,t) = 0$ and $U(x,t) \to 0$ uniformly in t as $x \to \infty$.

Taking Fourier cosine transform to given partial differential equation, we get

$$\Rightarrow \quad \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial U}{\partial t}\cos sx\;\,dx \;-\; s^2\,\bar{U}_c(s\,) - \left(\frac{\partial U}{\partial x}\right)_{x=0} + t\,\bar{U}_c(s\,) = 0$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\sqrt{\frac{2}{\pi}}\int_0^\infty U\;\cos sx\,dx \;=\; s^2\,\bar{U}_c(s\,) - t\,\bar{U}_c(s\,)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}F_c(U) = s^2\,\bar{U}_c(s\,) - t\,\bar{U}_c(s\,) \;\;\Rightarrow\;\; \frac{\partial}{\partial t}\bar{U}_s(s\,) \;=\; s^2\,\bar{U}_c(s\,) - t\,\bar{U}_c(s\,)$$

or $\frac{d\,\bar{U}_c}{dt} = (s^2 - t)\bar{U}_c$. This is 1st order DE, its solution is,

$$\therefore \quad \bar{U}_c = \bar{U}_c(s,t) = A\ e^{s^2 t - \frac{t^2}{2}}, \qquad \dots\ (1)$$

At $t = 0$, $\bar{U}_c = A$ $\qquad \dots\ (2)$

Taking Fourier cosine transform to the condition, we get

$$F_c\{U(x,0)\} = F_c\{e^{-x}\}, \quad \text{at} \quad t = 0$$

$$\Rightarrow \quad \bar{U}_c = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx\ dx = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\cos sx + s\, sinsx) \right]_0^\infty$$

$$\Rightarrow \quad \bar{U}_c = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}, \quad \text{at} \quad t = 0 \qquad \dots\ (3)$$

Equations (2) & (3) give, $A = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$, put this value in equation (1), we get

$$\bar{U}_c = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}\ e^{s^2 t - \frac{t^2}{2}} .$$

Taking inverse Fourier cosine transform, we obtain

$$U(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}\ e^{s^2 t - \frac{t^2}{2}} \right] \cos sx\, ds$$

$$\Rightarrow \quad U(x,t) = \frac{2}{\pi}\ e^{-\frac{t^2}{2}} \int_0^\infty \frac{1}{1+s^2}\ e^{s^2 t} \cos sx\ ds.$$

5. Conclusion

In this paper, the Laplace and Fourier transform method for solving partial differential equations is studied. Using the theorems and the properties of the transforms we solved the partial differential equations using both the transforms. We concluded that Laplace and Fourier transforms are the influential transforms to effect with these equations that the solution is obtained without finding the general solution.

References

[1] I. N. Sneddon, Use of Integral Transforms, Mc Graw-Hill book Co.

[2] Kreyszig E, Advanced Engineering Mathematics, John Willey and Son's, Inc. New York (1999).

[3] Goel & Gupta, Integral Transforms, Pragati Prakashan, Meerut, (2001).

[4] Dass H K, Advanced Engineering Mathematics, S. Chand & Co. Ltd. (2007).

[5] Lokenath D and Bhatta D, Integral Transforms and Their Applications, Third edition, Chapman & Hall/CRC (2015).

[6] Sukhen Bhattacharyya, Puneet Kumar, A Study of Laplace Transform and Applications to Differential Equations, International Journal of Multidisciplinary Studies(2019).

[7] V. I. Gishlarkaev, Fourier-Transform Method for Partial Differential Equations: Formulas for Representing Solutions to the Cauchy Problem, *Vestnik St.Petersb. Univ.Math.* 55, 301–312 (2022).

[8] Paul L. Butzer & Rolf J. Nessel , Fourier Transform Methods and Second-Order Partial Differential Equations Mathematische Reihe, vol 1. Birkhäuser Basel.

[9] D.H. Mugler, R.A. Scott, Fast Fourier transform method for partial differential equations, the 2-D diffusion equation, Computers & Mathematics with Applications, Volume 16, Issue 3,(1988).

[10] Aggarwal S and Bhatnagar K, Dualities between Laplace Transform and Some Useful Integral Transforms, IJEAT,(2019).

[11] Jafari H, A new general integral transform for solving integral equations, Journal of Advanced Research (2020).

[12] Aggarwal S and Chaudhary R, A Comparative Study of Mohand and Laplace transforms, JETIR, 6(2), 230-240, (2019).

[13] M. Elzaki [c], Mohamed Elbadri [a b], Mohamed Z. Mohamed [d], Solution of partial differential equations by new double integral transform (Laplace - Sumudu transform), Ain Shams Engineering Journal, Volume 12, Issue 4, December (2021).