IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

Analyzing The Existence Of A System For Nonlinear First Order Ordinary Differential Equations Using Banach Contraction Mapping Principle

[1]**G.Jyothi** [0000-0002-3182-1149], [2] **Dr. Tejaswini Pradhan**[0000-0001-7510-2640],

[1]Research Scholar, Department of Mathematics, Kalinga University,
Kotni, Naya Raipur, Chhattisgarh-492101, India
[2]Assistant Professor, Department of Mathematics, Kalinga University,
Kotni, Naya Raipur, Chhattisgarh-492101, India

ABSTRACT

This article is concerned with fixed point theorems and their use in solving differential equations of n^{th} order. In this work, we introduce fixed-point theory as a tool for studying stability theory for ordinary and functional differential equations. The challenges with Liapunov's direct approach to stability analysis inspired this article. It also takes several cases into account. Several previous findings in the literature are extended and generalized by our work. Picard's theorem, fixed points, Lipschitz maps, and nonlinear operators are some examples of what we mean by "keywords.".

Keywords: Fixed point, Banach fixed-point theorem, System of linear equations, Fredholm integral equation.

INTRODUCTION

Nonlinear functional equations are encountered in various branches of science. These equations model many complex phenomena. Most of the nonlinear equations do not have exact solutions, hence iterative/numerical methods have to be explored. Classical Picard method or a method of successive approximations is used to solve a first order nonlinear differential equation $y'(x) = f(x, y(x))$ with initial condition $y(0) = c$. Ramos used method of variation of parameters and a Lipschitz continuity condition to determine a nonlinear Volterra integral equation for the solution of a second order ODE and further presented Picard-Lindeloff iterative procedure. In the present paper we employ Banach contraction principle to solve various nonlinear functional equations including partial differential equations (PDEs), ordinary differential equations (ODEs), integral equations, fractional differential equations (FDEs), system of ODEs/FDEs, algebraic equations and so on. The results obtained are compared with exact solutions and/or with the results obtained by other methods such as Adomian decomposition method (ADM) and the new iterative method (NIM) proposed by Daftardar-Gejji and Jafari.

Definition 2.1 Riemann-Liouville fractional integration of order α is defined as

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} f(y)\, dy, \quad t > 0.$$

Definition 2.2 Caputo fractional derivative of order α is defined as

$$D_t^{\alpha} f(t) = I_t^{m-\alpha} \left(\frac{d^m f(t)}{dt^m} \right), \quad 0 \le m-1 < \alpha \le m.$$

Note that for $0 \le m - 1 < \alpha \le m$, $a \ge 0$ and $\gamma > -1$

Definition 2.3 Let X and Y be metric spaces and F be a mapping from X into Y. F is said to be Lipschitz if there exists a real number $r \ge 0$ such that for all x, y ∈ X we have $d(F_x, F_y) \le rd(x, y)$. F is said to be contraction if $r < 1$.

LITERATURE REVIEW

Ahmed Alsaedi (2022) In this study, we provide conditions under which solutions exist and are unique for a self-adjoint coupled system of nonlinear second-order ordinary differential equations with nonlocal integral multi-point coupled boundary conditions over an arbitrary domain. Leray-Schauder alternative and Schauder's fixed point theorem provide proofs of existence, whereas the Banach contraction mapping principle ensures that a unique solution exists. At last, a few examples are built to showcase the findings.

Ali Rezaiguia (2022) To that end, we focus on a specific kind of neutral nonlinear differential equations in this paper. This sort of issue has adequate circumstances for the existence of positive periodic solutions, and these criteria are derived using Krasnoselskii's fixed-point theorem.

Boutiara, (2022) In this work, we focus on a nonlinear fractional boundary value issue with a Caputo-Hadamard derivative and investigate whether or not solutions exist and are unique. Applying the notion of Banach contraction, we prove several novel findings about existence and uniqueness. Scheafer's and Krasnoselskii's fixed point theorems are used to derive additional existence results. Some examples are given to help illustrate the points later on.

Mannan, M. (2021) For mapping findings presented in the context of normed space, this work intends to provide a study of the Banach fixed point theorem. This study generalizes the standard Banach fixed point theorem. In order to explain why certain maps, provide optimal solutions under certain circumstances, a fixed-point theory has been developed as an elegant synthesis of many branches of mathematical analysis. Many subsequent mathematical conclusions, such as the Picard-Lindel of Theorem, the Picard theorem, and the implicit function theorem, rely on this fixed-point theory. We also came up with some ideas on how to simply deduce numerous well-known fixed-point theorems from the Banach theorem. It builds upon previous research that has extended the Banach contraction principle to metric and norm spaces.

Mardanov (2014) Here, we investigate the existence of solutions for boundary-value issues involving ordinary differential equations with two-point and integral boundary conditions. Using standard fixed-point theorems, we get conclusions of existence and uniqueness. It also includes several examples to help illustrate the points made.

PROBLEM FORMULATION

The Banach fixed point theorem is discussed next, which provides both necessary and sufficient criteria for the existence and uniqueness of a fixed point, as well as a constructive process for generating crisp results close to the fixed point. First, let's establish some terms.

Definition 1. An example of this would be defining a mapping T from a nonempty set X to itself. It is claimed that Xx is a fixed point of the mapping T if and only if.

$$Tx = x$$

to put it another way, x is a perfect match for the picture T x.

Definition 2. Let $X = (X, d)$possess the properties of a metric space. Some sort of charting $T : X \longrightarrow X$ is referred to as a Lipschitz map if the integer is finite. $c > 0$in so that everyone benefits $x, y \in X$

$$d(Tx, Ty) \leq cd(x, y)$$

If there exists a positive real integer c > 1 such that for any x, y X, then T is said to be a contraction on X.

Definition 3. So, we'll pretend that X is a metric space. Some kind of charting $T : X \rightarrow X$is weakly contractive on X if and only if

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for all $x, y \in X$ and $\varphi[0, \infty) \longrightarrow [0, \infty)$If a function that does not decrease in value over time, then brings down to $\varphi(t) = 0$ if and only if $t = 0$. Clearly, if $\varphi(t) = \kappa t$ where $0 < \kappa < 1$,

Proposition1. If X is a metric space and d is a dimension, then every contraction mapping on X is a continuous mapping.

Theorem 3. Let X be a metric space that isn't empty, then we'll prove the Banach fixed point theorem. Imagine X is finished and $T : X \longrightarrow X$has been shortened from X. There is thus just one fixed point in T. $x \in X$.

Remark Practically speaking, the mapping T is a contraction over a subset of X, rather than the whole of X. T has a fixed point on the closed subset if and only if there is a restriction on the choice of x_0 such that the x_n lie in the closed subset, since a closed subset of a complete space X is complete.

The following theorem provides proof of this.

Theorem 4 Let $X = (X, d)$be a whole metric space, and allow it to $T : X \longrightarrow X$resemble a closed ball in their shrunken form $\overline{B} = \{x : d(x, x_0) \leq r\} \; \forall \, x_0, x \in \overline{B} \subset X$.

And let's presume that

$$d(x_0, Tx_0) < (1 - c)r.$$

Therefore, there is exactly one fixed point Xx in the space T.

The remainder of this work will be devoted to demonstrating how the proofs of the Baire Category theorem may be modified to prove the existence and uniqueness of solutions to the vector differential equation (6).

MAIN RESULTS

Here, we present a Study of Banach Fixed Point Theorem and its Application's for mapping results which is introduced in setting of normed spaces such as.

3.1. Banach Contraction Theorem (or Principle)

Here we will give the proof of Banach contraction theorem (or principle) both for metric space and normed space separately.

Theorem-1: Let T be a contraction mapping on a complete metric space X. Then T has a unique fixed point.

Proof: Let us consider an arbitrary point $x \in x_0$ and define the iterative sequence (x_n) by

$$\left.\begin{aligned} & x_0, x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \cdots, x_n = Tx_{n-1} \\ & \text{Then, } x_2 = TTx_0 = T^2 x_0 \\ & \qquad x_3 = TT^2 x_0 = T^3 x_0 \\ & \qquad \vdots \\ & \qquad x_n = T^n x_0 \end{aligned}\right\}$$

Then the sequence of the image of x_0 under repeated application of T. We now show that

(x_n) is a cauchy sequence.

If n >m , then

$$d\left(x_{m+1}, x_m\right) = d\left(Tx_m, Tx_{m-1}\right)$$

$$\Rightarrow d\left(x_{m+1}, x_m\right) \leq Kd\left(x_m, x_{m-1}\right)$$

$$\Rightarrow d\left(x_{m+1}, x_m\right) \leq Kd\left(Tx_{m-1}, Tx_{m-2}\right)$$

$$\Rightarrow d\left(x_{m+1}, x_m\right) \leq K^2 d\left(x_{m-1}, x_{m-2}\right)$$

Proceeding in this way up to m times we get,

$$d\left(x_{m+1}, x_m\right) \leq K^m d\left(x_1, x_0\right)$$

Hence by the triangle inequality we obtain for n > m

$$\begin{aligned} d\left(x_m, x_n\right) &\leq d\left(x_m, x_{m+1}\right) + d\left(x_{m+1}, x_{m+2}\right) + \cdots + d\left(x_{n-1}, x_n\right) \\ &\leq K^m d\left(x_0, x_1\right) + K^{m+1} d\left(x_0, x_1\right) + \cdots + K^{n-1} d\left(x_0, x_1\right) \\ &= K^m \left(1 + k + \cdots + k^{n-m-1}\right) d\left(x_0, x_1\right) \\ &= k^m \frac{1 - k^{n-m}}{1 - K} d\left(x_0, x_1\right) \end{aligned}$$

Since 0 < 1 < K , So that the number $1 - k^{n-m} < 1$

$$d\left(x_m, x_n\right) \leq \frac{k^m}{1 - K} d\left(x_0, x_1\right)$$

Again dx (x_0 , x_1) is fixed and 0 < 1 < K , so we can make the right hand side as small as we please by taking m sufficiently large. This shows that (x_n) is a cauchy sequence.

Since X is complete, there exists a point $x \in X$ Such that $x_n \to x$. Now we show that this limit x is a fixed point of the mapping T. From triangle inequality and by definition we have

$$d(x,Tx) \le d(x,x_n) + d(x_n,Tx)$$

$$d(x,Tx) \le d(x,x_n) + Kd(x_{n-1},x)$$

We know that d (x,y) =0 if and only if x = y.

Conclusion

The Banach theorem has several obvious restrictions. Any continuous function translating the unit interval into itself must have a fixed point, as far as I can tell. This study was done with the intention of advancing functional analysis as it relates to normed spaces and fixed-point theory. The findings we get generalize previously known fixed-point conclusions in the context of Banach spaces to the spaces that are the norms of Banach spaces. If this research yields the predicted findings, we will have a much clearer grasp of how to solve the difficult theorem. In the future, we will talk about Banach spaces, specifically the features of its norm spaces and how they apply to a physical situation.

REFERENCE

1. Ahmed Alsaedi (2022),” Existence Theory for A Self-Adjoint Coupled System of Nonlinear Ordinary Differential Equations with Nonlocal Integral Multi-Point Boundary Conditions,” Applied Mathematics E-Notes, 22(2022), 496-508 c ISSN 1607-2510 Available free at mirror sites of http://www.math.nthu.edu.tw/amen/

2. Ali Rezaiguia (2022),”A Novel Implementation of Krasnoselskii’s Fixed-Point Theorem to a Class of Nonlinear Neutral Differential Equations,”Advances in Geometric Function Theory with Analytic Function Spaces Volume 2022 | Article ID 9242541 | https://doi.org/10.1155/2022/9242541

3. Boutiara, Abdelatif&Benbachir, Maamar&Guerbati, Kaddour. (2022). Existence and Uniqueness Solutions of a BVP for Nonlinear Caputo-Hadamard Fractional Differential Equation. Journal of Applied Nonlinear Dynamics. 11(2). 359-. 10.5890/JAND.2022.06.007.

4. Mannan, M., Rahman, M.,Akter, H., Nahar, N. and Mondal, S. (2021) A Study of Banach Fixed Point Theorem and It’s Applications. American Journal of Computational Mathematics, 11, 157-174. doi: 10.4236/ajcm.2021.112011.

5. Mardanov, Misir&Sharifov, Yagub&Molaei, Hamidreza. (2014). Existence and uniqueness of solutions for first-order nonlinear differential equations with two-point and integral boundary conditions. Electronic Journal of Differential Equations. 2014.

6. T. Bentrcia and A. Mennouni, “On the asymptotic stability of a Bresse system with two fractional damping terms: theoretical and numerical analysis,” Discrete and Continuous Dynamical Systems - Series B, 2022.

7. Mennouni and A. Youkana, “Finite time blow-up of solutions for a nonlinear system of fractional differential equations,” Electronic Journal of Differential Equations, vol. 152, pp. 1–15, 2017.

8. Ardjouni, A. Rezaiguia, and A. Djoudi, “Existence of positive periodic solutions for fourth order nonlinear neutral differential equations with variable delay,” Advances in Nonlinear Analysis, vol. 3, no. 3, pp. 157–163, 2014.

9. Ardjouni and A. Djoudi, “Existence of positive periodic solutions for a nonlinear neutral differential equation with variable delay,” Applied Mathematics E-Notes, vol. 12, pp. 94–101, 2012.

10. Ardjouni and A. Djoudi, “Existence of positive periodic solutions for two types of second-order nonlinear neutral differential equations with variable delay,” Proyecciones Journal of Mathematics, vol. 32, no. 4, pp. 377–391, 2013.

11. Ardjouni and A. Djoudi, “Stability in nonlinear neutral differential equations with variable delays using fixed point theory,” Electronic Journal of Differential Equations, vol. 128, no. 43, pp. 1–11, 2011.

12. X.-B. Shu, X. Fei, and Y. Shi, “S-asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations,” Applied Mathematics and Computation, vol. 270, pp. 768–776, 2015.

13. Y. Guo, M. Chen, X.-B. Shu, and F. Xu, “The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm,” Stochastic Analysis and Applications, vol. 39, no. 4, pp. 643–666, 2021.