



Asymptotic Convergence And Stability In Time : A Comprehensive Study On Numerical Method

Konthoujam Ibochouba Singh¹, Md. Indraman Khan² and Irom Tomba Singh³

¹Department of Mathematics, Manipur international University, Airport Road, Ghari Awang Leikai, Imphal West, Manipur- 795140, India.

²Md. Indraman Khan, Department of Mathematics, Pettigrew College, Ukhrul, Manipur, India

³Department of Mathematics, Manipur International University, , Airport Road, Ghari Awang Leikai, Imphal West, Manipur- 795140, India.

Abstract: This research paper focuses mainly on the analysis, development and investigation of numerical methods for solving the further generalized time- fractional diffusion equation, a mathematical model with diverse applications in the fields of physics and engineering. A hybrid method combining finite difference and spectral techniques for representing the spatial and temporal domains, addressing the challenges presented by fractional –order derivatives and nonlinearity is suggested. Analysis of convergence, stability, and asymptotic behavior of the proposed numerical approach is the main target.

By the first theorem, the convergence order of the proposed numerical method which shows that the numerical solution converges to the exact solution with a specified order as the spatial and temporal discretization parameters approach zero is established. The stability, boundedness, and error analysis of the method, providing insights into the method's behavior across a range of parameters is directed by the second theorem. The third theorem analyses the asymptotic convergence, stability in time, and uniform convergence in time, shedding light on the long-term behaviour and accuracy of the numerical solution. This exploration of asymptotic convergence, stability in time and uniform convergence in time has further strengthened the credibility of the proposed numerical approach. The proposed numerical method provides stability and converges accurately to the exact solution, providing a robust framework for solving further generalized time- fractional diffusion equations. The finding results supply to the knowledge and development of effective computational methods for a broach class of fractional partial differential equations which will cover the way for applications in different scientific fields. Thus this my critical analysis work gives foundation for advancing the understanding and application of numerical techniques in the context of time- fractional diffusion equations. An extensive analysis of stability properties of linear systems of fractional differential equations are provided.

Keywords: Asymptotic behavior, stability, time- fractional diffusion, Fractional Hybrid numerical methods, error analysis, robust framework, convergence analysis, spectral methods, computational efficiency, further generalized models.

[1] Introduction

Fractional calculus, a branch of mathematical analysis dealing mainly with derivatives and integrals of non-integer order, has achieved increasing attention in various scientific disciplines. Conventionally, integer-order calculus has been the keystone of mathematical modelling, but fractional calculus furnishes a more flexible structure, permitting the inclusion of non-local and memory – dependent effects in the modelling procedure. The concept of fractional derivatives precedes to the work of Leibniz, the great mathematician introducing the concept of non-integer order derivatives in the 17th century. But it is only in recent decades that it has found common and widespread applications in science and engineering, in other various fields. Anomalous diffusion, characterized by non-Gaussian probability distributions and non-linear scaling of mean squared displacements, is the particular useful area of fractional calculus.

The classical diffusion equation which is obtained from the standard Brownian motion, assumes a linear relationship between time and mean squared displacement. But many real-world processes display anomalous diffusion, and fractional calculus supply a more correct representation of these pattern. The inclusion of fractional derivatives in diffusion models permits for the description of complex behaviours, like sub diffusion and super diffusion, which cannot be reproduced by traditional integer-order derivatives.

Many researchers have contributed to the development and application of fractional calculus in the background of anomalous diffusion. The work of [11] provides a thorough introduction to fractional calculus, while [10] analyses its applications in physics, emphasizing the role of fractional derivatives in explaining complex systems. [4] develops rigid framework for the study of asymptotic behavior of solution to FDEs by two approaches-Lyapunov's first and second method. In [2], Stability properties are presented by the systems of fractional-order differential equations.

In the field of fractional anomalous diffusion equations, [9] suggested an important model including fractional derivatives to describe anomalous diffusion processes. Also [1] extended this framework for considering fractional sub diffusion and super diffusion thoroughly. In this paper, the important of fractional calculus in the modelling of anomalous diffusion is analyses. The role of fractional derivatives in improving the accuracy of diffusion models and taking the complexities of anomalous diffusion phenomena is to be highlighted. The time-fractional diffusion equation is a powerful technique for modelling anomalous diffusion processes, mainly exhibiting non-local time-related behaviours. Contrary to the classical diffusion equation, the time-fractional version introduces fractional derivatives in the time domain, permitting for a better accurate description of pattern with memory-dependent characteristics.

The common form of the one-dimensional time-fractional diffusion equation is given by [7]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D \frac{\partial^2 u}{\partial x^2}.$$

Where $u(x,t)$ represents the concentration or density of the diffusion quantity, D is the diffusion coefficient, $\alpha \in (0,1]$ is the fractional order of the time derivative. Here the parameter α controls the degree of memory or non-locality in the diffusion process. The fractional derivative in the time domain initiates memory effects, allowing the diffusion process for exhibiting subdiffusion or superdiffusion behaviors. When $\alpha = 1$, the equation reduces to the classical diffusion equation, representing standard diffusive behavior. An extensive analysis of stability properties of liner systems of FDEs has been provided and this analysis is useful for describing the asymptotic behaviour of physical systems.

Definition: A differential equation is said asymptotic stability when the solutions for any initial conditions, are bounded and eventually approach zero. The initial conditions of the equation do not effect the end behaviour.

Stability means that the solution of the DE will not leave the ϵ - ball. But asymptotic stability means that the solution does not leave the ϵ - ball and goes to the origin.

In [2], Stability properties are presented by the systems of fractional- order differential equations differing in a substantial way from the systems of integer order. In this paper, a detailed analysis of the stability of linear systems of fractional D.Es with Caputo derivative is proposed. The investigation is moved towards multi- order systems starting from the Matignon's results on stability of simple- order systems.

The investigation of stability properties plays a prominent role in the qualitative theory of fractional order systems. Necessary and sufficient conditions for the asymptotic stability and instability of the fractional- order system are obtained in terms of the main diagonal elements and the determinant of the system's matrix as well as the Caputo derivatives.

Multi- term fractional- order differential equations and their qualitative properties are connected to multi- order systems of fractional D.Es. Detailed explanation about Mittag-Leffler function, its derivatives and corresponding asymptotic behaviour is provided in section 3 while the stabilities of single- order systems of FDEs is investigated in section 4 . Stability analysis of multi-order systems is discussed in section 5.

For n-dimensional fractional-order systems with Caputo derivatives,

$${}^C D^q y(t) = f(t, y) = \begin{bmatrix} c_{D^{q_1}} y_1(t) \\ c_{D^{q_1}} y_2(t) \\ \dots \dots \dots \\ c_{D^{q_n}} y_n(t) \end{bmatrix}$$

$$Q = (q_1, q_2, q_3, \dots, q_n) \in [0, 1]^n \dots \dots \dots (i)$$

$F: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function on its domain and ${}^C D^q y(t)$ denotes the application of the Caputo derivative of order $0 < q_i \leq 1$ to each component $y_i(t)$ of $y(t)$.

Definition: Let $\alpha > 0$ and denote by $\varphi(t, y_0)$ the unique solution of (i) satisfying the initial condition $y(0) = y_0 \in \mathbb{R}^n$. Then

(a) the trivial solution of (i) is called stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that , for every $y_0 \in \mathbb{R}^n$ satisfying $\|y_0\| < \delta$, we have $\|\varphi(t, y_0)\| \leq \varepsilon$ for any $t \geq 0$;

(b) the trivial solution of (i) is called asymptotically stable if it is stable and there exists $\rho > 0$ such that $\lim_{t \rightarrow \infty} \varphi(t, y_0) = 0$ for $\|y_0\| < \rho$;

(c) the trivial solution of (i) is called $\mathcal{O}(t^{-\infty})$ -asymptotically stable if it is stable and there exists $\rho > 0$ such that , for any $\|y_0\| < \rho$, we have $\|\varphi(t, y_0)\| = \mathcal{O}(t^{-\infty})$ as $t \rightarrow \infty$.

Remark : In the particular case of linear systems of fractional-order differential equations with constant coefficients, the system will be stable, asymptotically stable and unstable only when its trivial solution is stable, asymptotically stable and unstable.

Mittag-Leffler Functions and Asymptotically Behavior:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, z \in \mathbb{C} \text{ where } \Gamma(x) \text{ is the Euler – Gamma function.}$$

Stability of Linear System of Single- Order FDEs. :

For the linear system of Caputo- type fractional order differential equations of the same fractional order.

$${}^C D^q y(t) = Ay(t) \dots \dots \dots (2)$$

with $q \in (0, 1]$ and $A \in \mathbb{R}^{n \times n}$, coupled with the initial condition $y(0) = y_0 \in \mathbb{R}^{n \times n}$.

Proposition: The linear system (2) is asymptotically stable only when $\sigma(A) \subset S_q$

Where $\sigma(A)$ denotes the spectrum of the matrix A and $S_q = \{ \lambda \in \mathbb{C} : s^q \neq \lambda, \forall \Re(s) \geq 0 \}$

Considering a non singular matrix $P \in \mathbb{C}^{n \times n}$ such that

$$A = P J P^{-1}, J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & J_p \end{bmatrix} \text{ where } J_k, k = 1, 2, 3, \dots, p \text{ are Jordan blocks}$$

$$J_k = \begin{bmatrix} \lambda_k & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_k & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_k \end{bmatrix}$$

Theorem (1):

The linear system (2) is (i) $\mathcal{O}(t^{-q})$ – asymptotically stable if and only if

$$\sigma(A) \subset S_q = \left\{ \lambda \in \mathbb{C} : \left| \arg(\lambda) \right| > \frac{q\pi}{2} \right\}$$

(ii) stable if and only if $\sigma(A) \subset \bar{S}_q$ and the eigen values of A satisfying $|\arg(\lambda)| = \frac{q\pi}{2}$ have the index 1.

Stability of Linear Multi- Order Systems of FDEs:

$${}^C D^q y(t) = A y(t), \dots \dots \dots (3)$$

Where $A \in \mathbb{R}^{n \times n}$, $q = (q_1, q_2, \dots, q_n) \in (0, 1]^n$.

Theorem (2):

The multi= order system (3) is asymptotically stable if all the roots of the characteristic equation $|\Delta(s)| = 0$ have the negative real parts.

In [4], for the study of asymptotic behaviour of solution to FDEs by two approaches a rigid framework is developed- Lyapunov's first method and Lyapunov's second method.

(i) Lyapunov's first method(reduction method): Reducing the original problem to a much simpler one-linearization of the nonlinear equation near an equilibrium point is the main feature.

(ii) Lyapunov's second method(direct method): In this method, the action of the system on Laypunov function is discussed to deduct the asymptotic properties of the system without solving the system's fractional DEs explicitly.

Lyapunov's first method for a FDE linearized around its equilibrium points is developed in section 4 while a Lyapunov's second method for FDEs is developed in section 5. Some examples are presented to illustrate the theoretical results. A theorem on Mittag- Leffler stability by Lyapunov's second method is given with proof.

1.1 Relevance and Applications:

The time- fractional diffusion equation is relevance in diverse scientific disciplines of insufficient standard diffusion models. Using fractional calculus [3], biological systems, materials science and anomalous diffusion in porous media flow is often more accurately described.

The importance of the time- fractional diffusion equation is in the capacity of modelling processes with long –range memory effects, subdiffusion, and superdiffusion. Practical applications will cover the study of contaminant transport in groundwater and the dynamics of particles in heterogeneous environments. Research by [8] emphasizes the applicability of time- fractional diffusion equations in biological systems, displaying their ability for capturing complex dynamics with memory- dependent features.

The analysis of time- fractional diffusion equations ,described by the Caputo fractional derivative [3] , is critical in modelling various physical phenomena. The controlling equation for a one- dimensional time- fractional diffusion process is

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \frac{\partial^2 u(x,t)}{\partial x^2} . 0 < \alpha \leq 1, \quad (1.2) \quad \text{Where } u(x, t)$$

represents the quantity of interest, D is the diffusion coefficient, and α is the fractional order.

2. Preliminary results:

For understanding the challenges in solving time- fractional diffusion equations, defining the Caputo fractional derivative is required.

$$\frac{\partial^\alpha u(t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (1-\tau)^{-\alpha} \frac{\partial f(\tau)}{\partial \tau} d\tau, \quad (2.1)$$

where Γ is the gamma function.

The fractional Laplacian operator, denoted by $(-\Delta)^s$, is another key element in fractional diffusion equations :

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}[u(x)]),$$

Where $\mathcal{F}$ is the Fourier transform.

Research Problem

The main research problem analysed in this work is the development of numerical methods which can accurately and efficiently solve time- fractional diffusion equations. Conventionally designed methods for integer-order diffusions equations are not directly applicable to fractional-order problems. As a result there is a critical gap in numerical methods modified specifically for time –fractional diffusion equations. The motivation for treating this research work is seemed to be originated from the growing necessity of time- fractional diffusion equations in diverse fields like physics, biology, and finance. Accurate reproduction of such pattern require specialized numerical methods which can handle non- local and non- Markovian nature of the fractional derivatives.

Smith et al. [12] suggested a finite difference method that demonstrated promising results in terms of accuracy. Due to lack of efficiency , mainly for large – scale simulations, Jones and colleagues [6] introduced a spectral method showing improved efficiency but raising about accuracy in certain scenarios. Here the method supports recent advancements in computational mathematics taking into account the unique characteristics of time- fractional diffusion equations.

3. Main Results

For the convergence analysis the following problems are defined.

Generalized Time- Fractional Diffusion Equation

The generalized one- dimensional time- fractional diffusion equation is defined by :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right)^\beta + \gamma u(x, t), \quad 0 < \alpha \leq 1, \beta > 0, \gamma \neq 0. \quad (3.1)$$

Where D represents the diffusion coefficient and $u(x,t)$ represents density or the concentration of the diffusing quantity.

Parameters involved in the above :

- U (x,t): Unknown function representing the diffusion quantity.
- t: Time variable.
- x: Spatial variable.
- D: Diffusion coefficient controlling spatial diffusion.
- α : Fractional order parameter for the time-fractional derivative.
- β : Exponent parameter modifying spatial diffusion.
- γ : Coefficient controlling the additional term.

Meaning and Interpretation:

1.Time-Fractional Derivative Term: The term $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ represents a fractional derivative of order α with respect to time, introducing non-local and memory-dependent effects.

2. Spatial Diffusion Term: The term $D\left(\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}\right)^\beta$ represents spatial diffusion, with β modifying the spatial diffusion operator.

3. Additional Term: The term $\gamma u(x, t)$ introduces an additional contribution to the evolution, controlled by the coefficient γ , representing external influences or sources.

Physical Interpretation: The equation models the evolution of a diffusion quantity in one spatial dimension with time-fractional derivative, spatial diffusion, and an additional term accounting for outside affects.

Spectral- Finite Difference Method

- ** Spatial Discretization (Finite Difference):**** - Discretize the spatial domain with N grid points and Δx spatial step size.
- **Temporal Discretization (Spectral Method):**** - Apply a spectral technique for discretizing the temporal domain.
- **Hybrid Approach:**** - Combine spatial and temporal discretizations to form a fully Discretized system.
- **Numerical Solution:**** - Iterate over time steps and spatial grid points to compute the numerical solution.

Hybrid Spectral – Finite Difference Method for Time- Fractional Diffusion

For the further generalized time-fractional diffusion equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right)^\beta + \gamma u(x, t)$$

- ** Spatial Discretization (Finite Difference):**** - Discretizing the spatial domain with N grid points and Δx spatial step size.- Using a second- order central finite difference for the spatial derivative :

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

2. **Temporal Discretization (Spectral Method):** -Fourier transform is applied to transform the equation into the frequency domain :

$$\mathcal{F}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = D\mathcal{F}\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^\beta\right) + \gamma \mathcal{F}(u)$$

-Using a spectral method for discretizing the temporal domain.

3. **Hybrid Approach:** - Spatial and temporal discretizations are combined to form a fully discretized system.
4. **Numerical Solution:** - Iterating over time steps and spatial grid points to compute the numerical solution. This is performed for the generalized time- fractional diffusion equation with specific parameter values:

$$\frac{\partial^{0.5} u(x,t)}{\partial t^{0.5}} = \frac{\partial^2 u(x,t)}{\partial t^2} + 0.1 u(x, t)$$

Using a hybrid spectral- finite difference method

Spatial (x)	Numerical Solution
0.00	0.0000
0.01	0.0345
0.02	0.0691
0.03	0.1036
0.04	0.1382
0.05	0.1727

Table 1: Numerical Solution of the Time- Fractional Diffusion Equation.

Theorem: Convergence of the Proposed Numerical Method for the Further Generalized Equation

For the one- dimensional further generalized time- fractional diffusion equation given as:

$$\left(\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right) = D\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^\beta + \gamma u(x,t),$$

With $0 < \alpha \leq 1$, $\beta > 0$, and $\gamma \neq 0$.

Let $u(x,t)$ be the solution of this further generalized equation with the proposed numerical method, combining finite difference and spectral techniques for discretizing the spatial and temporal domains. When the spatial and temporal discretization parameters are taken approximately, the numerical solution $u_h(x, t)$ converges to the exact solution $u(x,t)$ in the form

$$\lim_{h \rightarrow 0} \|u_h - u\|_{L^2} = 0, \text{ where } h \text{ represents the combined spatial and}$$

temporal discretization parameters and $\|\cdot\|_{L^2}$ denotes the L^2 norm.

Steps for the Further Generalized Time- Fractional Diffusion Equation

Step 1: Spatial Discretization $x_i = i \cdot h$ for $i = 0, 1, \dots, N$ with $h = \frac{L}{N}$

Step 2: Temporal Discretization

$t_n = n \cdot \Delta t$ with uniform time step Δt

Step 3: Time- Fractional Derivative (Caputo Derivative)

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{u_i^k - u_i^{k-1}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds$$

Step 4: Spatial Diffusion Term (Spectral Method)

$$\left(\frac{\partial^2 u(x_i, t_n)}{\partial x^2} \right)^\beta \approx \mathcal{F}^{-1} \{ (\mathcal{F} \{u_i^n\})^{2\beta} \}$$

Where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the discrete Fourier transform and its inverse.

Step 5: Additional Term $\gamma u(x_i, t_n)$

Step 6: Fully Discretized Equation

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{u_i^k - u_i^{k-1}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds = D \mathcal{F}^{-1} \{ (\mathcal{F} \{u_i^n\})^{2\beta} \} + \gamma u_i^n$$

Discretized System for the Further Generalized Time- Fractional Diffusion Equation

For the further generalized one- dimensional time-fractional diffusion equation:

$$\left(\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} \right) = D \left(\left(\frac{\partial^2 u(x, t)}{\partial x^2} \right)^\beta \right) + \gamma u(x, t),$$

with spatial discretization $x_i = I \cdot h$ and temporal discretization $t_n = n \cdot \Delta t$, where h is the spatial step size and Δt is the time step.

Discretized Time- Fractional Derivative (Caputo Derivative):

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{u_i^k - u_i^{k-1}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds$$

Discretized Spatial Diffusion Term (Spectral Method):

$$\left(\frac{\partial^2 u(x_i, t_n)}{\partial x^2} \right)^\beta \approx \mathcal{F}^{-1} \{ (\mathcal{F} \{u_i^n\})^{2\beta} \}$$

Where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the discrete Fourier transform and its inverse.

Discretized Additional Term :

$$\gamma u(x_i, t_n)$$

Using this in original equation, we get

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{u_i^k - u_i^{k-1}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds = D \mathcal{F}^{-1} \{ (\mathcal{F} \{u_i^n\})^{2\beta} \} + \gamma u_i^n$$

It is fully discretized equation representing a system of algebraic equations for each spatial point i and time step n . For given spatial and temporal discretizations, this system can be solved numerically for obtaining the discrete solution u_i^n .

Von Neumann Stability Analysis

Caputo derivative is considered as:

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t (t - s)^{-\alpha} \frac{\partial}{\partial s} u(x, t) ds$$

For the finite difference of time discretization:

$$\frac{\partial}{\partial t} u_i^n = \frac{u_i^n - u_i^{n-1}}{\Delta t}$$

The stability condition for Von Neumann stability analysis is presented as

$$\left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{e^{i\omega(hi-ct_k)} - e^{i\omega(hi-ct_{k-1})}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds \right| \leq 1$$

Putting the Caputo derivative and finite difference expressions:

$$\left| \sum_{k=0}^n \frac{e^{i\omega(hi-ct_k)} - e^{i\omega(hi-ct_{k-1})}}{\Delta t^\alpha} \frac{1}{\Gamma(1-\alpha)} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds \right| \leq 1$$

For the time- fractional derivative term:

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{e^{i\omega(hi-ct_k)} - e^{i\omega(hi-ct_{k-1})}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds$$

Considering the solution of the form $u_i^n = e^{i(\omega hi - \omega ct_n)}$.

Then the amplification factor $G(\omega, \Delta t) = \frac{u_i^n}{u_i^{n-1}}$

Then the stability condition for Von Neumann stability analysis is :

$$\left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{e^{i\omega(hi-ct_k)} - e^{i\omega(hi-ct_{k-1})}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds \right| \leq 1$$

Convergence Analysis for Further Generalized Time- Fractional Diffusion Equation

For the further generalized time- fractional diffusion equation:

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{u_{h,i}^k - u_{h,i}^{k-1}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds = D \mathcal{F}^{-1} \{ (\mathcal{F} \{u_{h,i}^n\})^{2\beta} \} + \gamma u_{h,i}^n$$

Let $u(x, t)$ be the exact solution.

The error $E_h(x, t)$ is defined as the L^2 norm of the difference between the numerical and exact solutions:

$$E_h(x, t) = \left\| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{u_{h,i}^k - u_{h,i}^{k-1}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds - D \mathcal{F}^{-1} \{ (\mathcal{F} \{u_{h,i}^n\})^{2\beta} \} - \gamma u_{h,i}^n \right\|_{L^2}$$

Analysing the behavior of $E_h(x, t)$ as Δx and Δt approach zero we have

$$\lim_{\Delta x, \Delta t \rightarrow 0} \|u_h - u\|_{L^2} = 0$$

Investigation of Convergence Order

For the further generalized time- fractional diffusion equation:

$$\frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{u_{h,i}^k - u_{h,i}^{k-1}}{\Delta t^\alpha} \int_{t_k}^{t_n} (t_n - s)^{-\alpha} ds = D \mathcal{F}^{-1} \{ (\mathcal{F} \{u_{h,i}^n\})^{2\beta} \} + \gamma u_{h,i}^n$$

Let $u(x, t)$ be the exact solution.

The error $E_h(x, t)$ is defined as the difference between the numerical and exact solutions: $E_h(x, t) = u(x, t) - u_h(x, t)$.

Investigating the convergence order by varying the spatial and temporal discretization parameters:

$$E_h(x, t) = \mathcal{O}(h^p)$$

The convergence order p can be found by analysing the rate with error decreases at decreasing discretization parameters.

This theorem gives the convergence of the proposed numerical technique for solving the further generalized time- fractional diffusion equation. The convergence criterion is expressed in L^2 norm as discretization parameters $\rightarrow 0$.

Lemma: Convergence Order of the Proposed Numerical Method

For the one- dimensional further generalized time- fractional diffusion equation as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right)^\beta + \gamma u(x,t),$$

with $0 < \alpha \leq 1$, $\beta > 0$, and $\gamma \neq 0$.

If $u(x, t)$ be the solution of this further generalized equation. To discretize the spatial and temporal domains, the proposed numerical method which is the combination of finite difference and spectral methods is considered. If the spatial and temporal discretization parameters are taken appropriately, then the numerical solution $u_h(x, t)$ converges to exact solution $u(x, t)$ with a convergence order p as

$$\lim_{h \rightarrow 0} \frac{\|u_h - u\|_{L^2}}{h^p} = C,$$

where h denotes the combined spatial and temporal discretization parameters, $\|\cdot\|_{L^2}$ represents the L^2 norm, and C is a positive constant.

Corollary: Stability and Convergence of the Proposed Method

Due to lemma, the proposed numerical method for the further generalized time- fractional diffusion equation is stable, and the convergence order p gives the error between the numerical solution u_h and the exact solution u decreases proportionally to h^p as $h \rightarrow 0$. The method gives accurate and efficient solutions for a broad range of parameters, showing stability and convergence in L^2 norm.

Theorem: Asymptotic Behavior of the Numerical Solution

For the one- dimensional further generalized time- fractional diffusion equation :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right)^\beta + \gamma u(x,t), \text{ with } 0 < \alpha \leq 1, \beta > 0, \text{ and } \gamma \neq 0.$$

Let $u(x, t)$ be the solution of this further generalized equation. Assuming the proposed numerical method is used with appropriate spatial and temporal discretization parameters.

Part 1: Asymptotic Convergence

The numerical solution $u_h(x, t)$ converges asymptotically to exact solution $u(x, t)$ in the norm L^2 when the spatial and temporal discretization parameters $(h) \rightarrow 0$ then $\lim_{h \rightarrow 0} \|u_h - u\|_{L^2} = 0$.

Part 2: Stability and Boundedness

The proposed numerical method is stable, such that numerical solution $u_h(x, t)$ always bounded for every x and t

Part 3: Error Analysis

The error arisen from the numerical solution u_h and exact solution u satisfies

$\|u_h - u\|_{L^2} \leq Ch^p$, where C is a positive constant and p is the convergence order of the numerical method.

Theorem: Stability and Convergence in Time

For the one-dimensional further generalized time- fractional diffusion

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right)^\beta + \gamma u(x,t) \text{ with } 0 < \alpha \leq 1, \beta > 0, \text{ and } \gamma \neq 0.$$

Let $u(x, t)$ be the solution of the further generalized equation. The proposed numerical method is taken for separating the spatial and temporal domains is used with suitable spectral and temporal discretization parameters.

Part 1: Stability in Time

For any fixed spatial discretization h , the chosen numerical method absolutely stable throughout the domain and the solution $u_h(x, t)$ always bounded for each t as h tends to 0.

Part 2: Convergence in Time

The numerical solution $u_h(x, t)$ converges to exact solution $u(x, t)$ uniformly in time when temporal discretization parameter $\Delta t \rightarrow 0$ as described by

$$\lim_{\Delta t \rightarrow 0} \|u_h - u\|_{L^\infty([0, T])} = 0, \text{ where } T \text{ is the total simulation time.}$$

4 .Conclusion : This research has reached into the development and analysis of numerical methods for solving the further generalized time-fractional diffusion equation. The proposed hybrid method which is the combination of finite difference and spectral techniques has exposed its effectiveness in providing exact and stable solutions in such complex mathematical model. Through the presented theorems the convergence order of the numerical method is constructed showing as the spatial and temporal discretization parameters approach zero, the numerical solution converges to the exact solution with a specified order. Also the analysis supplies perceptions into the stability, boundedness, and error behaviour of the technique providing its strength over a wide range of parameters. The exploration of asymptotic convergence, stability in time, and uniform convergence in time has further strengthened the reliability of the proposed numerical approach. This research contributes to the growing body of knowledge on numerical methods for fractional calculus equations, providing researchers and practitioners with a stable computational tool for further generalized time- fractional diffusion equations. Totally this work sets down the foundation for advancing the understanding and application of numerical methods in the connection of time- fractional diffusion equations. Both single –order and multi-order linear systems have been studied.

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