IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

Requirement Of Best Approximation Property involving Chebyshev Spline Function

Mr. Dharmendra Kumar Singh
Research Scholar
Ram Chandra Chandravanshi University
Bishrampur, Palamu (Jharkhand)
And
Dr. Ram Kumar Sinha
Associate Professor in Mathematics
Ram Chandra Chandravanshi University
Bishrampur, Palamu (Jharkhand)

Abstract

Given a point g and a set M in a Normed linear space, a point of M of minimum distance from g is called a best approximation and the problem of determining such a point is called best approximation problem.

Our Main aims are then to investigate important example of such problems, with particular regards of –

(1) The existence of best approximation.
(2) The uniqueness of best approximation.
(3) The characterization of best approximation.
(4) The construction of method for determining best approximation.

It is clear that the existence of a solution to approximation problems requires certain additional conditions to be satisfied.

We will not deal with this aspect of the problem to any great extent, but will be concentrated rather on the treatment of problems for which the normed linear space S, set M and point g are assumed given motivation for the use of any particular norm will however, be given where appropriate.

The task of characterizing best approximations is that of deriving potentially useful conditions which a best approximation must satisfy, and which, if satisfied guarantee that a particular approximation is indeed a best approximation. The importance of approximation problems have solution which is characterized by conditions which can be used directly as a means of actually calculating such best approximations. The construction of this solution by their means forms an important part of this paper and results are included in algorithms.

The aim ⟨1⟩ to ⟨3⟩ listed above can be useful treated in some general results are given in this paper.

Keywords:- Best approximation, Normed linear space. Existence, uniqueness, characterization.

Section-1

1.1. INTRODUCTION

In this paper we study the spline function and its application to the approximation theory to find the best approximation property by the large number of method available in the literature for the system order, reduction and approximation theory. We mention below a few of them:-

(1). Rough approximation method (RAM)
(2). Direct rough approximation method (DRAM)
(3). Based on rough approximation criterion method (BRACM)
(4). State space rough approximation method (SSRAM)
(5). New frequency domain technique method (NFDTM)
(6). Continued Differentiation method (CDM)
(7). Aggregation technique method (ATM)
(8). Singular perturbation method (SPM)
(9). System reduction by introducing time delay method (SRTTDM)
(10). Spline approximation problem method (SAPM)

In this paper we consider briefly and exclusively chebyshev approximation by spline function.
This does not imply by any means that there is a complete theory or even that this paper represents a survey of all that is known.
There remains many interesting question in spline approximation problem.

1.2. Definition of spline function

The term spline function was first used by SCHOENBERG in1946. It is suggested by the Mechanical device known as "Spline". Given a strictly increasing sequence of real numbers $x_0, x_1, \ldots \ldots \ldots . . x_{n-1}, x_n$, a simple spline function $S(x)$ of degree m with knots $x_1, \ldots \ldots \ldots . . x_{n-1}$ is a real valued function defined on $[x_0, x_n]$ such that it satisfies the following properties-

1. The restriction of S on $[x_i, x_{i+1}]$ for $i = 0,1,2,3, \ldots \ldots . . , n-1$ is given by some polynomial of degree m or less.
2. S and its derivatives of order 1,2,3,......,m-1 are continuous everywhere.

A cardinal spline is a spline function with knots as all the integers.

1.3. DEFINITION OF CHEBYSHEV SPLINE FUNCTION:-

A function $p(t)$ defined on $[a, b]$ is called Chebyshevian spline function of order n possessing the knots $x_1 x_2, \ldots . . . x_k$: $a < x_1 < x_2 < x_3 < \cdots \ldots . . < x_k < b$ with associated multiplicities.

$$\mu_1, \mu_2, \mu_3, \ldots \ldots . \mu_k \quad : \quad 1 \le \mu_i \le n;$$

$$i = 1,2, \ldots \ldots . . , k$$

Provided

1. In each of the intervals
$$[a, x_1), \quad [x_1, x_2) \; \ldots \; \ldots \; \ldots . \quad \ldots [x_k, b]$$
$P(t)$ coincides with a u −polynomial.
2. $P(t)$ is of continuity class
$$C^{n-\mu_i-1} \;\; at \; x_i \;, \qquad i = 1,2,. \; \ldots \ldots . . \;, k$$
when $\mu_i = n$, we interpret class C^{-1} as referring to function processing at worse jump discontinuities.

The following lemma represents a canonical representation of a Chebyshevian spline function with prescribed knots and multiplicities.

1.4. Advantage of spline functions.

Spline functions are most successful approximating functions for practical applications so far discovered one may be unaware of the fact that ordinary polynomials one in adequate in many situations. This particularly the case when one approximates functions which arise from physical world rather than from mathematical world. Function which express physical relationship are frequently of a disjointed or disassociated nature. That is to say, their behaviour in another region. Polynomial along with most other mathematical functions has just the opposite property. Namely, their behaviour in any small region determine their behaviour everywhere splines do not suffer this handicap yet for ≥ 3 , they represent nice smooth curves in the physical world. However, it is commonly observed that a polynomial of sufficiently high degree fitted to a fairly large number of given points exhibits more numerous and more serious undulations than a curve drawn with a spline.

1.5. Recent developments in the theory of splines

The spline approximation in its present form first appeared in a paper by SCHOENBERG in the year 1946. There is a very close relationship between spline theory and beam theory. SOKOLNIKOFF [1956, PP 1-4] provide a brief but very readable account of the development of beam theory. From the latter, one might anticipate some of the recent developments in the theory of spline, particularly the minimum curvature property. As suggested in SCHOENBERG'S PAPER [1946], approximations employed in actuarial work also frequently involve concepts the relate them closely to the spline.

After 1946, SCHOENBERG, together with some of his research students, continued these investigations of splines and Monosplines. In particular, SCHOENBERG and WHITENEY [1949-1953] first obtained criteria for the existence of certain spline of interpolation. For the case of splines of even order with interpolation at the junction points, a simpler approach to the question of existence due to AHLBERG, NILSON and WALSH [1964,1965] is now possible. It makes use of a basic integral relation obtained for cubic splines of interpolation to a function $f(x)$ on a mesh Δ by HOLLADAY [1957] which asserts

$$\int_a^b |f''(x)|^2 dx = \int_a^b |S''_{\Delta}(f,x)|^2 dx + \int_a^b |f''(x) - S''_{\Delta}(f,x)|^2 dx$$

Here $S_{\Delta}(f,x)$ denotes the spline of interpolation to f(x) on Δ.

1.6. Types of splines:

With respect to a mesh Δ:$a = t_0 < t_1 < \cdots \ldots \ldots \ldots \ldots \ldots . < t_n = b$ there are many varieties of splines on $[a,b]$ associated with a linear differential operator L.

A linear differential operator L defined by

$$L = a_n(t).D^n + a_{n-1}(t).D^{n-1} + \cdots + a_0(t)$$

where each $a_j(t)$ $(t = 0,1,\ldots.,n)$ is in $C^n[a,b]$, and $a_n(t)$ does not vanish on $[a,b]$.

Any particular spline $S_{\Delta}(t)$ denote the finite number of independent parameters which we have termed defining values. $S_{\Delta}(t)$ is the result of a particular assignment of parameters and each is allowed to vary over the field of real numbers, a $k - parameter$ family F_{Δ} of splines with respect to Δ results. The family F_{Δ} consists of splines differing only in the assignment of numerical values to these parameters and, consequently, consists of splines possessing similar continuity properties at mesh point. In this sense, F_{Δ} defines a type of spline.

Previously, for certain classes of splines, we have defined a cardinal spline as one for which all parameter values are zero except for one that is unity.

In this manner, with F_{Δ} we can associate k linearly independent splines $e_i(t)$ $(i = 1,2,3,\ldots..k)$, and F_{Δ} is a linear space with the $e_i(t)$ as a basis. As a consequence, if $S_{\Delta}(t)$ is in F_{Δ}, then

$$S_{\Delta}(t) = \sum_{i=1}^{k} a_i e_i(t).$$

If Δ: $a = x_0 < x_1 < \cdots \ldots \ldots \ldots \ldots \ldots . < x_N = b$ is a mesh on $[a, b]$ then a generalized spline of deficiency k $(0 \le k \le n)$ with respect to Δ is a function $S_{\Delta}(x)$ which is in $k\,[a, b]$ and satisfies the differential equation

$$L^* L S_{\Delta} = 0$$

on each open mesh interval of Δ.

where L^* be the formal adjoint of L; thus

$$L^* = (-1)^n . D^n \langle a_n(x) \rangle + (-1)^{n-1} . D^{n-1} \langle a_{n-1}(x) \rangle + \cdots \ldots \ldots \ldots \ldots \ldots \ldots . . -D \langle a_1(x) \rangle + a_0(x).$$

We also say that $S_{\Delta}(x)$ has order $2n$ when we want to indicate the order of the operator L^*L defining $S_{\Delta}(x)$.

If $K = 0$ and L has analytic coefficients, then $S_{\Delta}(x)$ has continuous derivatives of all orders and satisfies.

Throughout $[a, b]$ in this case, continuity of the $(2n-1)^{th}$ derivative implies continuity of the $2n^{th}$ and all higher derivatives.

The ordinary spline (deficiency one) allows discontinuities in the $(2n-1)^{th}$ derivative, but only at mesh points. In general, the deficiency of a spline is measure of the failure of the spline to satisfy on $[a, b]$.

<u>Section-2</u>

Chebyshev approximation by spline

2.1. Definition:-Let Q be a compact Housedroff space and denoted by $C(Q)$ the linear space of continuous real or complex valued functions on Q made into a Banach with the usual for $\|f\| = \max_{x \in Q} |f(x)|$ denotes the Chebyshev Norm. Approximation in this norm is called Chebyshev approximation or uniform approximation or Minimax approximation.
Least square approximation uses the norm on $L^2[a, b$ defined by $\|x\| = \langle x, x \rangle^{1/2} = \left(\int_a^b |x(t)|^2 dt\right)^{1/2}$

2.2. Existence of best approximation

Theorem-1

Let M denote a compact set in a Norm linear space. Then to each point g of the space there corresponds a point of M closest to g.

Proof:-

Let $\delta = \inf\{ \|g - x\|, x \in M \}$, it follows from the definition of inf that there exist a sequence of points $x_1, x_2, \ldots \ldots$ in M such that $\|g - x_n\| \to \delta$ as $n \to \infty$

Since M is compact, it follows that there exists a subsequence of $x_1, x_2, \ldots \ldots$, converging to $x^* \in M$. Now

$$\|g - x^*\| \le \|g - x_n\| * \|x_n - x^*\|$$

And so, letting $\to \infty$, $\|g - x^*\| \le \delta$ since the left hand side is independent of n, However, since $x^* \in M$.

$$\|g - x^*\| \ge \delta$$

Thus $\|g - x^*\| = \delta$ and x^* is a point of minimum distance from g in M.

While compactness of M is a sufficient condition for a best approximation to exist, it is clearly not necessary, for, consider the subset of the real line defined by the interval $[-x, 1]$.

This is not compact, yet the nearest point to any g exists. It is often possible to restrict the set of approximation to a compact subset of M, and thereby obtained the required result for any $g \geq \frac{1}{2}$, we could choose the best approximation from $[\frac{1}{2}, 1]$.

Theorem-2

If y is a finite dimensional subspace of a normed space $x = (x, \|.\|)$, then for each $x \in X$ there exists a best approximation to x out of Y.

Proof:

Here $x \in X$ be a given, consider the closed ball

$$\tilde{B} = (y \in Y : \|Y\| \leq 2\|X\|)$$

Then $o \in \tilde{B}$, so that for the distance from x to $\tilde{B}$. We obtained

$$\delta(x, \tilde{B}) = \inf_{\tilde{y} \in \tilde{B}} \|x - \tilde{y}\| \leq \|x - o\| = \|x\|$$

Now if $y \in \tilde{B}$, then $\|y\| > 2\|x\|$ and

$$\|x - y\| \geq \|y\| - \|x\| > 2\|x\| \geq \sigma(x, \tilde{B})$$

This show that $\delta(x, \tilde{B}) = \delta(x, Y) = \delta$, and this value can not be assumed by a $y \in Y - \tilde{B}$. Hence if a best approximation to x exists it must lie in $\tilde{B}$. We now see the reason for the use of $\tilde{B}$. Instead of the whole subspace Y may now consider the compact subset $\tilde{B}$, since $\tilde{B}$ is closed and bounded and Y is finite dimensional. The norm is continuous by above given theorem. Here $y_0 \in \tilde{B}$ such that $\|x - y\|$ assumes a minimum at $y = y_0$. By definition, y_0 is best approximation to x out of Y.

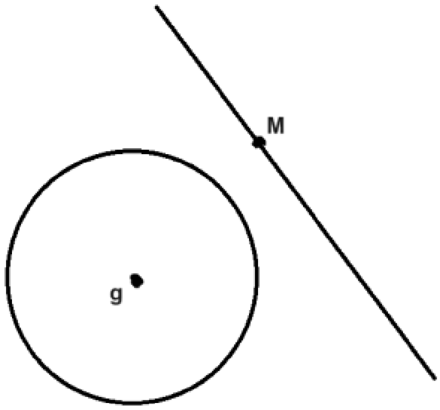
2.2.3. Uninqueness of best approximation

To find the best approximation to g from M may be confined to sets of the form

$$\{m : m \in M, \|m - g\| \leq r\}$$

Where r is constant.

For example, in the space R^2 equipped with the L_2 norm, the set $\|m - g\| \leq r$ just defines the interior and boundary of a circle, centre g and radius r, and geometrically, we are seeking the circle of minimum radius which passes through a point of M(See the given figure).



In general, the sets $\|m-g\| \leq r$ will be closed regions of different shapes, and the precise nature of the boundary would appear to play an important role in the question of uniqueness. This is in fact the case, and we begin the study of uniqueness of a closer examination of the properties of these closed regions.

Theorem:

In a strictly convex normed linear space S, a finite dimensional subspace M contains a unique best approximation to any point $g \in S$.

Proof:

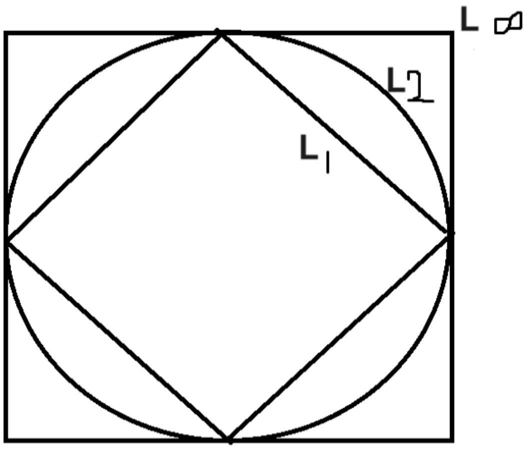
Let m_1 and m_2 be elements of M of minimum distance r from g. Then for $\emptyset < \lambda < 1$.

$$\|\lambda m_1 + (1-\lambda)m_2 - g\| \leq \|\lambda(m_1 - g)\| + \|(1-\lambda)(m_2 - g)\|$$

$$\leq r$$

So that $\|m_1 - g\| = \|m_2 - g\| = \|\lambda m_1 + (1-\lambda)m_2 - g\|$

This theorem shows that the uniqueness of the best approximation will be an immediate consequence of the strict convexity of the appropriate closed spheres. In particular, this information will be obtained by an examination of the unit balls (obtained by setting $= \emptyset, r = 1$). It is convenient to illustrate the case of the space R^2, equipped with the different L_p *norms* $1 \leq p \leq \infty$, and some of the uinit balls are shown in the figure.



The illustration show that R^2 with L_1 or L_∞ norms is not a strictly convex normed linear space, and suggested that R^2 with the L_p norms, $1 < p < \infty$. This is indeed the case, and in fact these properties of the unit balls carry over to the space R^n and C[x]

(the space of continuous real-valued functions defined on x, where x is for example an $n-$ $dimensional\ continuum$) with the L_p norms, $1 \leq p \leq \infty$.

2.2.4. Characterization of best approximation

In this section we give a general characterization result for an important class of linear approximation problems. Which includes all problems which may be expected to arise in practice, and all these which are dealt with in detail later on? The result is infact obtained by specializing form a wider class of non-linear problems, for which necessary condition for a best approximation are obtained. The unified nature of such results is therefore demonstrated, and particular theorems required in subsequent chapters will be shown to be easily obtained as special cases of this theory. (For a completely general treatment of the linear case, see SINGER,1970). The analysis required here of necessity draws on certain concepts and results from functional analysis, and readers without the necessary background may omit this section. The basic requirement here is a (possibly non linear) mapping $f(a)$ from R^n into a subset M of an arbitrary normed linear space S. Then, without loss of generality, the problem of finding a best approximation from M to an element of S can be stated. Find $a \in R^n$ to minimize $\|f(a)\|$.

Section-3 Conclusion

Our main aims are then to investigate important examples of such problems, with particular regard to:-

(i) The existence of best approximation
(ii) The uniqueness of best approximation
(iii) The characterization of best approximation
(iv) The construction of method for determining the best
(v) approximation

References:

1. Numerical mathematics and Computing, 5th Eddition, Thomas Indian Edition, Brooks/Cole, ISBN 81-315-0125-6, First Indian reprint 2007.
2. Kincaid, Addition Wesly , Cheneyiw, Numerical Analysis Mathematics of scientific computing Pacific Grove, Califroni, Brook/Cole, 3rd edition 2002.
3. Allen. M.S: Frequency-domain identification of linear time periodic system using LTI techniques. J Comput. Nonlinear Dyn. 4, 041,0041-041.004.6 (2009) Springer.
4. Sinha, S.C Chou, C.C, Denman, HH.: Stability analysis of system with periodic coefficients: an approximation approach. J Sound Vib. 64. 515-527 (1979). Doi:10.1016/0022-460X(79)90801-0
5. An approximation theory and approximation practice. SSIAM, Philadelphia (2013)
6. Sinha, R K: A study of certain aspects of approximation theory " A Doctoral thesis, Department of Mathematics. R.I.T. Jamshedpur (INDIA) Year. 1991-95.
7. R.L. Burden, J. Douglas Faaires, Young stwon state university, Numerical Analysis, 7th edition Indian edition, Brook/Cole, ISBN: 81-315-0137-X, Second Indian Reprint 2007.
8. Rice, J.R Characterization of Chebyshev approximation by splines. Siam. J.Num. Analysis. Vol-4. 1976, PP557-565.
9. Schoenberg, L L .: Spline Functions.: Basic Theory.Wiley, Hoboken (1981).
10. Kohuapt, L.: Differential calculus for p-norms of complex-valued fuctions with application. J. Comput. Appl. Math. 145(2). 425-457 (2002). doi.: 10.1016/S0377-0427(00)00559-8