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Derivatives And Fractional Integrals For Natural Transformation

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Abstract- In this paper, we provide the analysis of the natural transformation for certain distribution spaces in terms of the fractional integral and differential operators of the transform. There are two parts to this composition. The first is a brief introduction to fractional operators and the less researched and well-known natural transform. The basis for the research study is the understanding that the fractional integral and the Laplace and Sumudu transforms may be expressed as one of the appropriate representations of the Abel integral problem.

Keywords-Derivatives, Natural transformation, Integrals, Equation

Introduction

Laplace and Sumudu transformations are closely linked to the natural transform. Khan and Khan (1) introduced the natural transform, often known as the N-transform, and (2, 5) studied its characteristics. Maxwell's equations (3,4) were solved using the natural transform; more research on the natural transform might be explored. The natural transform often works with functions that are continuous and continuously differentiable, or if we suppose that the function is both continuous and fractionally derivative. The natural transform does not work as well as the Laplace and Sumudu transforms, however, since the function is not derivative. Therefore, in a same vein, we must redefine "natural transform." The integral transform approach has several applications in a wide range of technological and scientific fields. Ordinary and partial differential equations, which are often derived from physical occurrences, may be resolved using the integral transform approach. This is the basic principle that motivates researchers to create new integral transformations that are used to solve a number of problems in applied mathematics. The natural transform (N-transform), a novel integral transform, was recently introduced by academics, who also looked at its properties and its applications (1). Later, several of the properties and applications of natural transforms were discussed, and the inverse natural transform was defined (1, 2). The powerful analytical tools of the distribution theory may be used to tackle many problems that arise in the applied sector. Consequently, the various integral transformations to the distribution space (6, 7, 8, 9, 10, 11, 12, 13, and 14) are defined. The objective of this study is to extend the Natural transform in the distributional space with compact support and investigate some properties and theorems of the generalized integral transform.

Natural Transform

With reference to the articles [13] [14], the basic definitions of natural transform and its properties are introduced as follows:

$$(I_{a+}^{\alpha}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_a^{\infty}(x-t)^{\alpha-1}\varphi(t)dt \text{ , } x > a \tag{1}$$

$$(I_{b-}^{\alpha}\phi)(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{b}(t-x)^{\alpha-1}\phi(t)dt \text{ , } x < b, \tag{2}$$

where 0 > α (α being the order). These integrals are also known as the Riemann-Liouville fractional integrals or the left - sided and right - sided fractional integrals, respectively. The integrals given in (1) and (2) are extensions to half and (or) whole axis finite interval [a, b]. These may be used on the half axis) (a, ∞) or (−∞, b) respectively, subject to the variable limit of integration. For the half axis, we write

$$(I_{0+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^{x}(x-t)^{\alpha-1}\varphi(t)dt \quad , \ 0 < x < \infty \tag{3}$$

and on the whole real axis, it is given by

$$(I_{+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}\varphi(t)dt \quad , \ -\infty < x < \infty \tag{4}$$

and

$$(I_{-}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}\varphi(t)dt \quad , \ -\infty < x < \infty \tag{5}$$

Fractional derivatives of order α,0< α <1 are also called Riemann-Liouville fractional derivatives or the left - handed and right - handed fractional derivatives, respectively, in the interval [a, b].

Natural Transform of Fractional Derivative

If N [f (t)] is the natural transform of the function f (t), then the natural transform of fractional derivative of order α is defined as:

$$N\left[f^{(\alpha)}(t)\right]=\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0) \qquad (6)$$

Let the function f (t) belongs to set A be multiplied with weight function $e^{\pm t}$ then,

$$N\left[e^{\pm t}f(t)\right]=\frac{s}{s\mp u}R\left[\frac{s}{s\mp u}\right]$$

Let the function f (at) belongs to set A, where a is non zero constant then,

$$N\left[f(at)\right]=\frac{1}{a}R\left[\frac{s}{a},u\right]$$

If $w^n(t)$ is given by

$$w^n(t)=\int_0^t\cdots\int_0^t f(t)(\mathrm{d}t)^n\,\mathrm{d}t.$$

Then the natural transform of $w^n(t)$ is

$$N\left[w^n(t)\right]=\frac{u^n}{s^n}R(s,u)$$

Theorem 2.1 If $R_f(s,u)\triangleq \mathrm{N}\left[f(t)\right]$ for (s u) $\in\Omega_f$ then $R_f(s,u)$ is analytic on Ω_f .

Proof: Let (s, u) be arbitrary but fixed point in

$$\Omega_f=\left\{(u,s)\,\omega_1<\mathrm{Re}\left(\frac{s}{u}\right)<\omega_2\right\}$$

Choose the real positive number a,b and r such that

$$\omega_1<a\,\mathrm{Re}\left(\frac{s}{u}-r\right)<\mathrm{Re}\left(\frac{s}{u}+r\right)<b<\omega_2$$

Let ΔS be the complex increment such that $|\Delta S|<r$, and as

$\Delta S \neq 0$ we have

$$\frac{R_f(s+\Delta S,u)-R_f(s,u)}{\Delta S} - < f(t), \frac{\partial}{\partial s}\frac{1}{u}e^{\frac{-st}{u}} > \\ = < f(t), \psi\Delta S(t) >$$

$$\text{Where } \psi\Delta S(t) = \frac{1}{\Delta S}\left[e^{\frac{-(s+\Delta S)t}{u}} - e^{\frac{-st}{u}}\right] - \frac{\partial}{\partial s}\frac{1}{u}e^{\frac{-st}{u}}$$

$$(-D_t)^k \psi\Delta S(t) = \frac{1}{\Delta S}\left[\left(\frac{s+\Delta S}{u}\right)^k e^{\frac{-(s+\Delta S)t}{u}} - \left(\frac{s}{u}\right)^k e^{\frac{-st}{u}}\right] - \frac{\partial}{\partial s}\left(\frac{s}{u}\right)^k \frac{1}{u}e^{\frac{-st}{u}}$$

$$= \frac{1}{\Delta S 2\Pi i}\int_C \left(\left[\frac{1}{\xi-\left(\frac{s+\Delta S}{u}\right)} - \frac{1}{\xi-\left(\frac{s}{u}\right)}\right]\xi^k e^{\frac{-\xi t}{u}} d\xi - \frac{1}{2\Pi i}\int_C \frac{\xi^k e^{\frac{-\xi t}{u}}}{\xi-\left(\frac{s}{u}\right)^2} d\xi\right)$$

$$= \frac{\Delta S}{2\Pi i}\int_C \frac{\xi^k e^{\frac{-\xi t}{u}}}{\xi-\left(\frac{s+\Delta S}{u}\right)\xi-\left(\frac{s}{u}\right)^2} d\xi$$

Now for all $\xi \in C$ and $-\infty < t < \infty, \left|K_{a,b}(t)\xi^k e^{\frac{-\xi t}{u}}\right| \leq M$

where M is constant independent of ξ and t. Moreover

$$\xi - \left(\frac{s+\Delta s}{u}\right) > r_1 - r > 0 \text{ and } \left|\xi - \left(\frac{s}{u}\right)\right| = r_1$$

$$\left|K_{a,b}(t)D^k(t)\psi\Delta S\right| \leq \frac{|\Delta S|}{2\Pi}\int_C \frac{M}{(r_1-r)r_1^2}|d\xi| \\ \leq \frac{|\Delta S| M}{(r_1-r)r_1^2}$$

The RHS is independent of t and converges to zero as $|\Delta S| \to 0$.

This shows that $\psi_{\Delta S}$ converges to zero in $\mathcal{D}_{a,b}$ as $|\Delta S| \to 0$ which completes the proof of theorem.

Similar proof can be made for another variable u.

Conclusion

In this article, we defined the generalized natural transform and extended the natural transform in the distributional space with compact support. The inversion and analyticity theorems are shown. This work may open up new avenues for the study of generalized integral transforms. The Sumudu transform defined for a particular test function space can also be considered for the Schwartz space (those with similar properties), since the Riemann-Liouville fractional integral can be expressed as one of the forms of the Abel integral equation and the solution obtained is one of the fractional derivatives.

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