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# Minimax Multi objective Fractional Programming with B-(p, r)-Invexity with Non differentiable functions 

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#### Abstract

In this paper, we derive some theorems and duality theorems on non differentiable Multiobjective Fractional Minimax Programming Under B-(p, r)-Invexity functions.


Keywords: B-(p, r)-invex function and Duality.

## Introduction

The necessary and sufficient conditions for generalized minimax programming were first
developed by Schmitendorf [1]. Tanimoto [9] applied these optimality conditions to define a dual problem and derived duality theorems. Bector and Bhatia relaxed the convexity assumptions in the sufficient optimality condition in [1] and also employed the optimality conditions to construct several dual models which involve pseudo-convex and quasi-convex functions, and derived weak and strong duality theorems. Yadav and Mukherjee [16] established the optimality conditions to construct the two dual problems and derived duality theorems for differentiable fractional minimimax programming.

Convexity plays an important role in deriving sufficient conditions and duality for non linear programming problems Karush-Kuhn-Tucker type sufficient optimality conditions for nonlinear programming problems. Generalized invexity and duality for multiobjective programming problems are discussed in [5], and inseparable Hilbert spaces are studied by Soleimani-Damaneh [3]. Soleimani-Damaneh [4] provides a family of linear infinite problems or linear semi-infinite problems to characterize the optimality of nonlinear optimization problems. Recently, optimality conditions for a class of generalized fractional minimax programming problems involving $B(p, r)$ invexity functions and established duality theorems for various duality models.

This paper is organized as follows. In section 2, we give some preliminaries. An example which is B-(1, 1)-invex but convex. In section 3, we establish the sufficient optimality conditions. Duality results are presented in section4 by developing some theorems and duality theorems in non differentiable minimax multi objective fractional programming with $\mathrm{B}-(\mathrm{p}, \mathrm{r})$-invexity.

## Notations and Preliminaries

Definition: 1. Let $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{R}$ ( where $\mathrm{X} \subseteq \mathrm{R}^{\mathrm{n}}$ ) be differentiable function, and let $p$, $r$ be arbitrary real numbers. Then $f$ is said to be ( $p, r$ )-invex (strictly ( $p, r$ )-invex) with respect to $\eta$ at $u \in X$ on $X$ if there exists a function $\eta: X \times X \rightarrow R^{n}$ such that, for all $X \in X$, the inequalities.

$$
\begin{aligned}
& \frac{1}{r} e^{r(f(x))} \geq \frac{1}{r} e^{r(f(u))}\left[1+\frac{r}{p} \nabla f(u)\left(e^{p \eta(x, u)}-1\right)\right](>\text { if } x \neq u) \text { for } p \neq 0, r \neq 0 \\
& f(x)-f(u) \geq \frac{1}{p} \nabla f(u)\left(e^{p \eta(x, u)}-1\right)(>\text { if } x \neq u) \text { for } p \neq 0, r=0 \quad f(x)-f(u) \geq \nabla f(u) \eta(x, u) \\
& (>\text { if } x \neq u) \text { for } p=0, r=0 \text { hold. }
\end{aligned}
$$

Definition 2 : The differentiable function $f: X \rightarrow R$ (where $X \subseteq R^{n}$ ) is said to be (strictly) B-(p, $r$ )-invex with respect to $\eta$ and $b$ at $u \in X$ on $X$ if there exists a function $\eta: X \times X \rightarrow R^{n}$ and a function $\mathrm{b}: \mathrm{X} \times \mathrm{X} \rightarrow \mathrm{R}_{+}$such that, for all $\mathrm{X} \in \mathrm{X}$, the following inequalities.

$$
\begin{aligned}
& \frac{1}{r} b(x, u)\left(e^{r(f(x)-f(u))}-1\right) \geq \frac{1}{p} \nabla f(u)\left(e^{p n(x, u)}-1\right)(>\text { if } x \neq u) \text { for } p \neq 0, r \neq 0 \\
& \frac{1}{r} b(x, u)\left(e^{r(f(x)-f(u))}-1\right) \geq \nabla f(u) \eta(x, u)(>\text { if } x \neq u) \text { for } p=0, r=0
\end{aligned}
$$

$$
\mathrm{b}(\mathrm{x}, \mathrm{u})(\mathrm{f}(\mathrm{x})-\mathrm{f}(\mathrm{u})) \geq \frac{1}{\mathrm{p}} \nabla \mathrm{f}(\mathrm{u})\left(\mathrm{e}^{\mathrm{p}(\mathrm{n}(\mathrm{x}, \mathrm{u})}-1\right)(>\text { if } \mathrm{x} \neq \mathrm{u}) \text { for } \mathrm{p} \neq 0, \mathrm{r}=0
$$

$\mathrm{b}(\mathrm{x}, \mathrm{u})(\mathrm{f}(\mathrm{x})-\mathrm{f}(\mathrm{u})) \geq \nabla \mathrm{f}(\mathrm{u}) \eta(\mathrm{x}, \mathrm{u})(>$ if $\mathrm{x} \neq \mathrm{u})$ for $\mathrm{p}=0, \mathrm{r}=0$ hold. f is said to be (strictly) $B-(p, r)$-invex with respect to $\eta$ and $b$ on $X$ if it is $B-(p, r)$-invex with respect to same $\eta$ and $b$ at each $\mathrm{u} \in \mathrm{X}$ on X .

In this paper, we consider the following non differentiable minimax fractional programming
problem:

$$
\min _{x \in R^{n}} \sup _{y \in Y} \frac{l_{i}(x, y)+\left(x^{T} D x\right)^{\frac{1}{2}}}{m_{i}(x, y)-\left(x^{T} E x\right)^{\frac{1}{2}}}
$$

Subject to $\mathrm{g}(\mathrm{x}) \leq 0, \quad \mathrm{x} \in \mathrm{X}$
where Y is a compact subset of $R^{m}, l(\cdot, \cdot): R^{n} \times R^{m} \rightarrow R, m(\cdot, \cdot): R^{n} \times R^{m} \rightarrow R$ and $\mathrm{C}^{1}$ functions on $R^{n} \times R^{m}$ and $g(\cdot): R^{n} \times R^{p}$ is $\mathrm{C}^{1}$ function on $\mathrm{R}^{\mathrm{n}}, \mathrm{D}$ and E are $n \times n$ positive semidefinite matrices.

Let $S=\{x \in X: g(x) \leq 0\}$ denote the set of all feasible solutions of (FP)

Any point $x \in S$ is called the feasible point of (FP)

For each $(x, y) \in R^{n} \times R_{m}$, we define

$$
\Phi(x, y)=\frac{l_{i}(x, y)+\left(x^{T} D x\right)^{\frac{1}{2}}}{m_{i}(x, y)-\left(x^{T} E x\right)^{\frac{1}{2}}}
$$

such that for each $(x, y) \in S \times Y$,
$l_{i}(x, y)+\left(x^{T} D x\right)^{\frac{1}{2}} \geq 0$ and $m_{i}(x, y)-\left(x^{T} E x\right)^{\frac{1}{2}}>0$

For each $x \in S$, we define
$H(x)=\left\{h \in H: g_{h}(x)=0\right\}$,

$k(x)=\left\{(s, t, \bar{y}) \in N \times R_{+}^{s} X_{R}^{m s}: 1 \leq s \leq n+1, t=\left(t_{1}, t_{2}, \ldots . . t_{s+}\right) \in R_{+}^{s}\right.$
with $\left.\sum_{i=1}^{s} t_{i}=1, \bar{y}=\left(\bar{y}_{1}, \bar{y}_{2}, \ldots \ldots ., \bar{y}_{s}\right) \bar{y}_{i} \in Y(x)(i=1,2, \ldots, s)\right\}$

Since $l_{i}$ and $m_{i}$ are continuously differentiable and Y is compact in $\mathrm{R}^{m}$, it follows that for each $x^{*} \in s, Y\left(x^{*}\right) \neq \phi$, and for any $\bar{y}_{i} \in Y\left(x^{*}\right)$, we have a positive constant.

$$
k_{0}=\phi\left(x^{*}, \bar{y}_{i}\right)=\frac{l_{i}\left(x^{*}, \bar{y}_{i}\right)+\left(x^{* T} D x^{*}\right)^{\frac{1}{2}}}{m_{i}^{*}\left(x^{*}, \bar{y}_{i}\right)-\left(x^{* T} E x^{*}\right)^{\frac{1}{2}}}
$$

## Generalized Schwartz Inequality

Let A be a positive-semi definite matrix of order n . Then, for all $x, w \in R^{n}$,

$$
x^{T} A w \leq\left(x^{T} A x\right)^{\frac{1}{2}}\left(w^{T} A w\right)^{\frac{1}{2}}
$$

Equality holds if for some $\lambda \geq 0$

$$
A x=\lambda A w
$$

Evidently, if $\left(w^{T} A w\right)^{\frac{1}{2}} \leq 1$, we have

$$
x^{T} A w \leq\left(x^{T} A x\right)^{\frac{1}{2}}
$$

If the functions $l_{i}, g_{i}$ and $m_{i}$ in problem (FP) are continuously differentiable with respect to $x \in R^{n}$, then Lai et al [52] derived the following necessary conditions for optimality of (FP).

## Theorem 1 (Necessary Conditions):

If $x^{*}$ is a solution of (FP) satisfying $x^{* T} D x^{*}>0, x^{*^{T}} E x^{*}>0$ and $\nabla g_{h}\left(x^{*}\right), h \in H\left(x^{*}\right)$ are linearly independent, then there exist $\left(s, t^{*}, \bar{y}\right) \in k\left(x^{*}\right), k_{0} \in R_{+}, w, v \in R^{n}$ and $\mu^{*} \in R_{+}^{p}$ such that

$$
\begin{equation*}
\sum_{i=1}^{s} t_{i}^{*}\left\{\nabla l_{i}\left(x^{*}, \bar{y}_{i}\right)\right\}+D w-k_{0}\left\{\nabla m_{i}\left(x^{*}, \bar{y}_{i}\right)-E V\right\}+\nabla \sum_{h=1}^{p} t_{i}^{*} \mu_{h}^{*} g_{h}\left(x^{*}\right)=0 \tag{2}
\end{equation*}
$$

$l_{i}\left(x^{*}, \bar{y}_{i}\right)+\left(x^{*^{T}} D x^{*}\right)^{\frac{1}{2}}-k_{0}\left(m\left(x^{*}, \bar{y}_{i}\right)-\left(x^{* T} E x^{*}\right)^{\frac{1}{2}}\right)=0, i=1,2, \ldots, s$

$w^{T} D w \leq 1, v^{T} E V \leq 1$,
$\left(x^{*^{T}} D x^{*}\right)^{\frac{1}{2}}=x^{* T^{T}} D w$,
(6)

## Sufficient Conditions:

Under smooth conditions say, convexity and generalized convexity as well as differentiability, optimality conditions for these problems have been studied in the past few years. The intrinsic presence of nonsmoothness (the necessity to deal with nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings) is one of the most characteristic features of modern variational analysis. The optimality conditions for nondifferentiable multiobjective case by Kim et al [49]. Now, we prove the sufficient condition for optimality of (Fp) under the assumptions $\mathrm{B}-(\mathrm{p}, \mathrm{r})$-invexity.

## Theorem 2 (Sufficient condition):

Let $\mathrm{x}^{*}$ be a feasible of (FP) and there exist a positive integer $s, 1 \leq s \leq n+1, t^{*} \in R_{+}^{s}$, $\bar{y}_{i} \in Y\left(x^{*}\right)(i=1,2, \ldots, s), k_{0} \in R_{+}, w, v \in R^{n}$ and $\mu^{*} \in R_{+}^{p}$, Assume that
(i) $\sum_{i=1}^{s} t_{i}^{*}\left(l_{i}\left(\cdot, \bar{y}_{i}\right)+(\cdot)^{T} D w-k_{0}\left(m_{i}\left(\cdot, \bar{y}_{i}\right)\right)-(\cdot)^{T} E V\right) i s B-(p, r)$-invex at $x^{*}$ on s with respect
to $\eta$ and b satisfying $b\left(x, x^{*}\right)>0$ for all $x \in s$.
(ii) $\sum_{h=1}^{p} \mu_{h}^{*} g_{h}(\cdot)$ is $B_{g}-(p, r)$-invex at $x^{*}$ on $s$ with respect to the same function $\eta$, and with respect to the function $b_{g}$, not necessarily, equal to $b$.

Then $x^{*}$ is an optimal solution of (FP).

Proof: Suppose to the contrary that $\mathrm{x}^{*}$ is not an optimal solution of (FP). Then there exists an $\bar{x} \in S$ such that
$\operatorname{Sup}_{y \in Y} \frac{l_{i}(\bar{x}, y)+\left(\bar{x}^{T} D \bar{x}\right)^{\frac{1}{2}}}{m_{i}(\bar{x}, y)-\left(\bar{x}^{T} E \bar{x}\right)^{\frac{1}{2}}}<\operatorname{Sup}_{y \in Y} \frac{l_{i}\left(x^{*}, y\right)+\left(x^{* T} D x^{*}\right)^{\frac{1}{2}}}{m_{i}\left(x^{*}, y\right)-\left(x^{*^{T}} E x^{*}\right)^{\frac{1}{2}}}$

We note that
$\operatorname{Sup}_{y \in Y}=\frac{l_{i}\left(x^{*}, y\right)+\left(x^{* T} D x^{*}\right)^{\frac{1}{2}}}{m_{i}\left(x^{*}, y\right)-\left(x^{* T} E x^{*}\right)^{\frac{1}{2}}} \frac{l_{i}\left(x^{*}, \bar{y}_{i}\right)+\left(x^{* T} D x^{*}\right)^{\frac{1}{2}}}{m_{i}\left(x^{*}, \bar{y}_{i}\right)-\left(x^{*_{T}} E x^{*}\right)^{\frac{1}{2}}}=k_{0}$,
for $\bar{y}_{i} \in Y\left(x^{*}\right), i=1,2, \ldots \ldots s$ and
$\frac{l_{i}\left(\bar{x}, \bar{y}_{i}\right)+\left(\bar{x}^{T} D \bar{x}\right)^{\frac{1}{2}}}{m_{i}\left(\bar{x}, \bar{y}_{i}\right)-\left(\bar{x}^{T} E \bar{x}\right)^{\frac{1}{2}}} \leq \operatorname{Sup}_{y \in Y} \frac{l_{i}(\bar{x}, y)+\left(\bar{x}^{T} D \bar{x}\right)^{\frac{1}{2}}}{m_{i}(\bar{x}, y)-\left(\bar{x}^{T} E \bar{x}\right)^{\frac{1}{2}}}$

Thus, we have

$$
\frac{l_{i}\left(\bar{x}, \bar{y}_{i}\right)+\left(\bar{x}^{T} D \bar{x}\right)^{\frac{1}{2}}}{m_{i}\left(\bar{x}, \bar{y}_{i}\right)-\left(\bar{x}^{T} E \bar{x}\right)^{\frac{1}{2}}}<k_{0}, \text { for } i=1,2, \ldots . s
$$

it follows that

$$
l_{i}\left(\bar{x}, \bar{y}_{i}\right)+\left(\bar{x}^{T} D \bar{x}\right)^{\frac{1}{2}}-k_{0}\left(m_{i}\left(\bar{x}, \bar{y}_{i}\right)-\left(\bar{x}^{T} E \bar{x}\right)^{\frac{1}{2}}\right)<0, \text { for } i=1,2, \ldots . s
$$

From (1), (3), (5), (6) and (7), we obtain

$$
\sum_{i=1}^{s} t_{i}^{*}\left\{l_{i}\left(\bar{x}, \bar{y}_{i}\right)+\bar{x}^{T} D w-k_{0}\left(m_{i}\left(\bar{x}, \bar{y}_{i}\right)-\bar{x}^{T} E V\right)\right\}
$$

$$
\begin{aligned}
& \leq \sum_{i=1}^{s} t_{i}^{*}\left\{l_{i}\left(\bar{x}, \bar{y}_{i}\right)+\left(\bar{x}^{T} D \bar{x}\right)^{\frac{1}{2}}-k_{0}\left(m_{i}\left(\bar{x}, \bar{y}_{i}\right)-\left(\bar{x}^{T} E \bar{x}\right)^{\frac{1}{2}}\right)\right\} \\
& <0=\sum_{i=1}^{s} t_{i}^{*}\left\{l_{i}\left(x^{*}, \bar{y}_{i}\right)+\left(x^{* T} D x^{*}\right)^{\frac{1}{2}}-k_{0}\left(m_{i}\left(x^{*}, \bar{y}_{i}\right)-\left(x^{* T} E x^{*}\right)^{\frac{1}{2}}\right)\right\} \\
& =\sum_{i=1}^{s} t_{i}^{*}\left\{l_{i}\left(x^{*}, \bar{y}_{i}\right)+x^{* T} D w-k_{0}\left(m_{i}\left(x^{*}, \bar{y}_{i}\right)-\left(x^{* T} E V\right)\right)\right\}
\end{aligned}
$$

It follows that

$$
\begin{align*}
& \sum_{i=1}^{s} t_{i}^{*}\left\{l_{l}\left(\bar{x}, \bar{y}_{i}\right)+\bar{x}^{T} D w-k_{0}\left(m_{i}\left(\bar{x}, \bar{y}_{i}\right)-\bar{x}^{T} E V\right)\right\} \\
& <\sum_{i=1}^{s} t_{i}^{*}\left\{l_{i}\left(x^{*}, \bar{y}_{i}\right)+x^{* T} D w-k_{0}\left(m_{i}\left(x^{*}, \bar{y}_{i}\right)-x^{* T} E V\right)\right\} \tag{8}
\end{align*}
$$

As $\sum_{i=1}^{s} t_{i}^{*}\left(l_{i}\left(\cdot, \bar{y}_{i}\right)+(\cdot)^{T} D w-k_{0} m_{i}\left(\cdot, \bar{y}_{i}\right)-(\cdot)^{T} E V\right)$ is B-(p,r)-invex at $\mathrm{x}^{*}$ on s with respect to $\eta$ and $b$, we have

$$
\begin{aligned}
& \frac{1}{r} b\left(x, x^{*}\right)\left(e^{r}\left[\sum_{i=1}^{s} t_{i}^{*}\left(l_{i}\left(x, \bar{y}_{i}\right)+x^{T} D w-k_{0}\left(m_{i}\left(x, \bar{y}_{i}\right)-x^{T} E V\right)\right)\right]\right. \\
& \left.-\sum_{i=1}^{s} t_{i}^{*}\left(l_{i}\left(x^{*}, \bar{y}_{i}\right)+x^{* T} D w-k_{0}\left(m\left(x^{*}, \bar{y}_{i}\right)-x^{* T} E V\right)-1\right)\right\} \\
& \geq \frac{1}{p}\left\{\sum_{i=1}^{s} t_{i}^{*}\left(\nabla l_{i}\left(x^{*}, \bar{y}_{i}\right)\right)+D w-k_{0}\left(\nabla m_{i}\left(x^{*}, \bar{y}_{i}\right)-E V\right)\right\}\left\{e^{p \eta\left(x, x^{*}\right)}-1\right\}
\end{aligned}
$$

holds for all $x \in s$, and so for $x=\bar{x}$. Using (10.8) and $b\left(\bar{x}, x^{*}\right)>0$ together with the inequality above, we get
$\frac{1}{p}\left\{\sum_{i=1}^{s} t_{i}^{*}\left(\nabla l_{i}\left(x^{*}, \bar{y}_{i}\right)\right)+D w-k_{0}\left(\nabla m_{i}\left(x^{*}, \bar{y}_{i}\right)-E V\right)\right\}\left\{e^{p \eta\left(\bar{x}, x^{*}\right)}-1\right\}<0$

From the feasibility of $\bar{x}$ together with $\mu_{h}^{*} \geq 0, h \in H$, we have

$$
\begin{equation*}
\sum_{h=1}^{s} \mu_{h}^{*} g_{h}(\bar{x}) \leq 0 \tag{10}
\end{equation*}
$$

By $B g-(p, r)$-invexity of $\sum_{h=1}^{s} \mu_{h}^{*} g_{h}(\cdot)$ at $x^{*}$ on $s$ with respect to the same function $\eta$, and with respect to the function $b_{g}$, we have

$$
\frac{1}{r} b_{g}\left(\bar{x}, x^{*}\right)\left\{e^{r}\left[\sum_{h=1}^{p} \mu_{h}^{*} g_{h}(\bar{x})-\sum_{h=1}^{p} \mu_{h}^{*} g_{h}\left(x^{*}\right)\right]-1\right\} \quad \geq \frac{1}{p} \sum_{h=1}^{p} \nabla \mu_{h}^{*} g_{h}\left(x^{*}\right)\left\{e^{p \eta\left(\bar{x}, x^{*}\right)}-1\right\}
$$

Since $\left(x, x^{*}\right) \geq 0$ for all $x \in S$ then by (4) and (10), we obtain

$$
\begin{equation*}
\frac{1}{p} \sum_{h=1}^{p} \nabla \mu_{h}^{*} g_{h}\left(x^{*}\right)\left\{e^{p \eta\left(\bar{x}, x^{*}\right)}-1\right\} \leq 0 \tag{11}
\end{equation*}
$$

By adding the inequalities (9) and (11) we have

$$
\begin{array}{r}
\frac{1}{p}\left\{\sum_{i=1}^{s} t_{i}^{*}\left(\nabla l_{i}\left(x^{*}, \bar{y}_{i}\right)\right)+D w-k_{0}\left(\nabla m_{i}\left(x^{*}, \bar{y}_{i}\right)-E V\right)+\sum_{h=1}^{p} \nabla \mu_{h}^{*} g_{h}\left(x^{*}\right)\right\} \\
\left\{e^{p \eta\left(\bar{x}, x^{*}\right)}-1\right\}<0
\end{array}
$$

Which contradicts (2). Hence the result.

## 4. Duality Result:

In this section, we consider the following dual to (Fp).

where $H_{1}(s, t, \bar{y})$ denotes the set of all

$$
(a, \mu, k, v, w) \in R^{n} \times R_{+}^{p} R_{+} \times R^{n} R^{n}
$$

Satisfying

$$
\begin{align*}
& \sum_{i=1}^{s} t_{i}\left\{\nabla l_{i}\left(a, \bar{y}_{i}\right)+D w-k\left(\nabla m_{i}\left(a, \bar{y}_{i}\right)-E V\right)+\nabla \sum_{h=1}^{p} \mu_{h} g_{h}(a)\right\}=0 \\
& (12)  \tag{13}\\
& \sum_{i=1}^{s} t_{i}\left\{l_{i}\left(a, \bar{y}_{i}\right)+a^{T} D w-k\left(m_{i}\left(a, \bar{y}_{i}\right)-a^{T} E V\right)\right\} \geq 0
\end{align*}
$$

$$
\begin{equation*}
\sum_{h=1}^{p} \mu_{h} g_{h}(a) \geq 0 \tag{14}
\end{equation*}
$$

$$
\begin{equation*}
(s, t, \bar{y}) \in k(a) \tag{15}
\end{equation*}
$$

$$
\begin{equation*}
w^{T} D w \leq 1, \quad V^{T} E V \leq 1 \tag{16}
\end{equation*}
$$

If, for a triplet $(s, t, \bar{y}) \in k(a)$, the set $H_{1}(s, t, \bar{y})=\phi$, then we define the supremum over it to be $-\infty$. For convenience, we let,

$$
\Psi_{1}(\cdot)=\sum_{i=1}^{s} t_{i}\left(l_{i}\left(\cdot, \bar{y}_{i}\right)+(\cdot)^{T} D w-k\left(m_{i}\left(\cdot, \bar{y}_{i}\right)-(\cdot)^{T} E V\right)\right)
$$

Let $S_{F D}$ denote a set of all feasible solutions of problem (FD). Moreover, let $S_{1}$ denote

$$
\left.S_{1}=\left\{a \in R^{n}:(a, \mu, k, v, w, s, t, \bar{y}) \in S_{F D}\right)\right\}
$$

Now we derive the following weak, strong and strict converse duality theorems.

## Theorem 3 (Weak Duality):

Let x be a feasible solution of $(\mathrm{p})$ and $(a, \mu, k, v, w, s, t, \bar{y})$ be a feasible of (FD). Let
(1) $\sum_{i=1}^{s} t_{i}^{*}\left(l_{i}\left(\cdot, \bar{y}_{i}\right)+(\cdot)^{T} D w-k_{1}\left(m_{i}\left(\cdot, \bar{y}_{i}\right)-(\cdot)^{T} E V\right)\right)$ is B-(p,r)-invex at a on SUS ${ }_{1}$ with respect to $\eta$ and b satisfying $b(x, a)>0, \sum_{h=1}^{p} \mu_{h} g_{h}(\cdot)$ is $B_{g}-(p, r)$-invex at a on SUS ${ }_{1}$ with respect to the same function $\eta$ and with respect to the function $b_{g}$ not necessarily, equal to $b$.

Then

$$
\begin{equation*}
\operatorname{Sup}_{y \in Y} \frac{l_{i}(x, y)+\left(x^{T} D x\right)^{\frac{1}{2}}}{m_{i}(x, y)-\left(x^{T} E x\right)^{\frac{1}{2}}} \geq k \tag{17}
\end{equation*}
$$

Proof: Suppose to the contrary that

$$
\operatorname{Sup}_{y \in Y} \frac{l_{i}(x, y)+\left(x^{T} D x\right)^{\frac{1}{2}}}{m_{i}(x, y)-\left(x^{T} E x\right)^{\frac{1}{2}}}<k
$$

Then, we have
$l_{i}\left(x, \bar{y}_{i}\right)+\left(x^{T} D x\right)^{\frac{1}{2}}-k\left(m_{i}\left(x, \bar{y}_{i}\right)-\left(x^{T} E x\right)^{\frac{1}{2}}\right) \leq 0$, for all $\bar{y}_{i} \in Y$.

It follows from (10.5) that

$$
\begin{equation*}
t_{i}\left\{l_{i}\left(x, \bar{y}_{i}\right)+\left(x^{T} D x\right)^{\frac{1}{2}}-k\left(m_{i}\left(x, \bar{y}_{i}\right)-\left(x^{T} E x\right)^{\frac{1}{2}}\right)\right\} \leq 0 \tag{18}
\end{equation*}
$$

with at least one strict inequality, since

$$
t=\left(t_{1}, t_{2}, \ldots \ldots, t_{s}\right) \neq 0
$$

From (1), (13), (16) and (18), we have

$$
\Psi_{1}(x)=\sum_{i=1}^{s} t_{i}\left\{l_{i}\left(x, \bar{y}_{i}\right)+x^{T} D w-k\left(m_{i}\left(x, \bar{y}_{i}\right)-x^{T} E V\right)\right\}
$$

$$
\leq \sum_{i=1}^{s} t_{i}\left\{l_{i}\left(x, \bar{y}_{i}\right)+\left(x^{T} D x\right)^{\frac{1}{2}}-k\left(m_{i}\left(x, \bar{y}_{i}\right)-\left(x^{T} E x\right)^{\frac{1}{2}}\right)\right\}
$$

$$
<0 \leq \sum_{i=1}^{s} t_{i}\left\{l_{i}\left(a, \bar{y}_{i}\right)+a^{T} D w-k\left(m_{i}\left(a, \bar{y}_{i}\right)-a^{T} E V\right)\right\}
$$

$$
=\Psi_{1}(a)
$$

Hence

$$
\begin{equation*}
\Psi_{1}(x)<\Psi_{1}(a) \tag{19}
\end{equation*}
$$

Since $\sum_{i=1}^{s} t_{i}\left(l_{i}\left(\cdot, \bar{y}_{i}\right)+(\cdot)^{T} D w-k\left(m_{i}\left(\cdot, \bar{y}_{i}\right)-(\cdot)^{T} E V\right)\right)$ is B-(p, r)-invex at a on SUS ${ }_{1}$ with
respect to $\eta$ and b , we have

$$
\begin{aligned}
& \frac{1}{r} b(x, a)\left\{e ^ { r } \left[\sum_{i=1}^{s} t_{i} l_{i}\left(x, \bar{y}_{i}\right)+x^{T} D w-k\left(m_{i}\left(x, \bar{y}_{i}\right)-x^{T} E V\right)-\sum_{i=1}^{s} t_{i}\left(l_{i}\left(a, \bar{y}_{i}\right)\right)\right.\right. \\
& \left.\left.+a^{T} D w-k\left(m_{i}\left(a, \bar{y}_{i}\right)-a^{T} E V\right)\right]-1\right\}
\end{aligned}
$$

From (19) and $b(x, a)>0$ together with the inequality above,

## We get

$$
\frac{1}{p}\left[\sum_{i=1}^{s} t_{i}\left\{\nabla l_{i}\left(a, \bar{y}_{i}\right)+D w-k\left(\nabla m_{i}\left(a, \bar{y}_{i}\right)-E V\right)\right\}\right]\left(e^{p \eta(x, a)}-1\right)<0
$$

Using the feasibility of x -together with $\mu_{h} \geq 0, \quad h \in H$, we obtain

$$
\begin{equation*}
\sum_{h=1}^{p} \mu_{h} g_{h}(x) \leq 0 \tag{21}
\end{equation*}
$$

From hypothesis (ii) we have

$$
\frac{1}{r} b_{g}(x, a)\left\{e^{r}\left[\sum_{h=1}^{p} \mu_{h} g_{h}(x)-\sum_{h=1}^{p} \mu_{h} g_{h}(a)\right]\right\}_{-1} \geq \frac{1}{p} \sum_{h=1}^{p} \nabla \mu_{h} g_{h}(a)\left\{e^{p \eta(x, a)}-1\right\}
$$

$b_{g}(x, a) \geq 0$ then by (10.14) and (10.21), we obtain

$$
\begin{equation*}
\frac{1}{p} \sum_{h=1}^{p} \nabla \mu_{h} g_{h}(a)\left\{e^{p \eta(x, a)}-1\right\} \leq 0 \tag{22}
\end{equation*}
$$

Thus, by (20) and (22), we obtain the inequality
$\frac{1}{p}\left\{\sum_{i=1}^{s} t_{i}\left(\nabla l_{i}\left(a, \bar{y}_{i}\right)+D w-k\left(\nabla m_{i}\left(a, \bar{y}_{i}\right)-E V\right)\right)+\sum_{h=1}^{p} \nabla \mu_{h} g_{h}(a)\right\}\left\{e^{p \eta(x, a)}-1\right\}<0$
which contradicts (12). Hence (17) holds.

## Theorem 4 (Strong Duality):

Let $x^{*}$ be an optimal solution of (FP) and $\nabla g_{h}\left(x^{*}\right), h \in H\left(x^{*}\right)$ are linearly independent. Then there exist $\left(s, t^{*}, \bar{y}\right) \in k\left(x^{*}\right)$, and $\left(\mathrm{x}^{*}, \bar{\mu}, \overline{\mathrm{k}}, \overline{\mathrm{v}}, \overline{\mathrm{w}}\right) \in \mathrm{H}_{1}\left(\overline{\mathrm{~s}}, \overline{\mathrm{t}}, \overline{\mathrm{y}}^{*}\right)$ such that $\left(\mathrm{x}^{*}, \bar{\mu}, \overline{\mathrm{k}}, \overline{\mathrm{v}}, \overline{\mathrm{w}}, \overline{\mathrm{s}}, \overline{\mathrm{t}}, \overline{\mathrm{y}}^{*}\right)$ is a feasible solution of (FD). Further the hypotheses of weak duality theorem are satisfied for all feasible solutions ( $a, \mu, k, v, w, s, t, \bar{y}$ ) of (FD), then $\left(\mathrm{x}^{*}, \bar{\mu}, \overline{\mathrm{k}}, \overline{\mathrm{v}}, \overline{\mathrm{w}}, \overline{\mathrm{s}}, \overline{\mathrm{t}}, \overline{\mathrm{y}}^{*}\right)$ is an optimal of (FD), and the two objectives have the same optimal values.

Proof: If $\mathrm{x}^{*}$ be an optimal solution of (FP) and $\nabla g_{h}\left(x^{*}\right), h \in H\left(x^{*}\right)$ is linearly independent, then by Theorem 1 , there exist $\left(s, t^{*}, \bar{y}\right) \in k\left(x^{*}\right)$ and $\left(\mathrm{x}^{*}, \bar{\mu}, \overline{\mathrm{k}}, \overline{\mathrm{v}}, \overline{\mathrm{w}}\right) \in \mathrm{H}_{1}\left(\overline{\mathrm{~s}}, \overline{\mathrm{t}}, \overline{\mathrm{y}}^{*}\right)$ such that $\left(\mathrm{x}^{*}, \bar{\mu}, \overline{\mathrm{k}}, \overline{\mathrm{v}}, \overline{\mathrm{w}}, \overline{\mathrm{s}}, \overline{\mathrm{t}}, \overline{\mathrm{y}}^{*}\right)$ is feasible for (FD) and problems (FP) and (FD) have the same objective values and

$$
\bar{K}=\frac{l_{i}\left(x^{*}, \bar{y}^{*}\right)+\left(x^{*^{T}} D x^{*}\right)^{\frac{1}{2}}}{m_{i}\left(x^{*}, \bar{y}^{*}\right)-\left(x^{* T} E x^{*}\right)^{\frac{1}{2}}}
$$

The optimality of this feasible solution for (FD) thus follows from Theorem 3.

## Theorem 5 (Strict Converse Duality):

Let $x^{*}$ and $\left(\bar{a}, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^{*}\right)$ be the optimal solutions of (FP) and (FD), respectively, and $\nabla g_{h}\left(x^{*}\right), h \in H\left(x^{*}\right)$ is linearly independent. Suppose that $\sum_{i=1}^{s} t_{i}^{*}\left(l_{i}\left(\cdot, \bar{y}_{i}\right)+(\cdot)^{T} D w-\bar{k}\left(m_{i}\left(\cdot, \bar{y}_{i}\right)-(\cdot)^{T} E V\right)\right)$ is strictly B-(p, r)-invex at a on SUS $_{1}$ with respect to $\eta$ and b satisfying $b(x, a)>0$ for all $\mathrm{x} \in \mathrm{s}$. Furthermore, assume that $\sum_{\mathrm{h}=1}^{\mathrm{p}} \mu_{\mathrm{h}} \mathrm{g}_{\mathrm{h}}(\cdot)$ is $\mathrm{Bg}_{\mathrm{g}}$ $(\mathrm{p}, \mathrm{r})$-invex at a on $\operatorname{SUS}_{1}$ with respect to the same function $\eta$ and with respect to the function $\mathrm{b}_{\mathrm{g}}$, but not necessarily, equal to the function $b$. Then $x^{*}=\bar{a}$, that is, $\bar{a}$ is an optimal point in (FP) and

$$
\operatorname{Sup}_{y \in Y} \frac{l_{i}\left(\bar{a}, \bar{y}^{*}\right)+\left(\bar{a}^{T} D \bar{a}\right)^{\frac{1}{2}}}{m_{i}\left(\bar{a}, \bar{y}^{*}\right)-\left(\bar{a}^{T} E \bar{a}\right)^{\frac{1}{2}}}=\bar{K}
$$

Proof: We shall assume that $x^{*} \neq \bar{a}$, and reach a contradiction fom the strong duality theorm (theorm 4) it follws that that is, $\bar{a}$ is an optimal point in (FP) and

$$
\begin{equation*}
\operatorname{Sup}_{y \in Y} \frac{l_{i}\left(x^{*}, \bar{y}^{*}\right) \pm\left(x^{* T} D x^{*}\right)^{\frac{1}{2}}}{m_{i}\left(x^{*}, \bar{y}^{*}\right)-\left(x^{*^{T}} E x^{*}\right)^{\frac{1}{2}}}=\bar{K} \tag{23}
\end{equation*}
$$

By feasibility of $x^{*}$ together with $\mu_{h} \geq 0, h \in H$, we obtain

$$
\begin{equation*}
\sum_{h=1}^{p} \mu_{h} g_{h}\left(x^{*}\right) \leq 0 \tag{24}
\end{equation*}
$$

By assumption, $\sum_{h=1}^{p} \mu_{h} g_{h}(\cdot)$ is $\mathrm{Bg}_{\mathrm{g}}-(\mathrm{p}, \mathrm{r})$-invex at a on $\mathrm{SUS}_{1}$ with respect to $\eta$ and with respect
to the $\mathrm{b}_{\mathrm{g}}$. Then, by Definition (12), there exists a function by such that $b_{g}(x, a) \geq 0$ for all $x \in s$ and $a \in s_{1}$. Hence by (14) and (24).

$$
\frac{1}{r} b_{g}\left(x^{*}, \bar{a}\right)\left\{e^{r}\left[\sum_{h=1}^{p} \mu_{h} g_{h}\left(x^{*}\right)-\sum_{h=1}^{p} \mu_{h} g_{h}(\bar{a})\right]\right\}_{-1} \leq 0
$$

Then, from Definition (10.2), we get

$$
\begin{equation*}
\frac{1}{p} \sum_{h=1}^{p} \nabla \mu_{h} g_{h}(\bar{a})\left\{e^{p \eta}\left(x^{*}, a\right)-1\right\} \leq 0 \tag{25}
\end{equation*}
$$

Therefore, by (10.25), we obtain the inequality

$$
\frac{1}{p}\left\{\sum_{i=1}^{s} t_{i}\left(\nabla l_{i}\left(\bar{a}, \bar{y}_{i}\right)+D w-\bar{k}\left(\nabla m_{i}\left(\bar{a}, \bar{y}_{i}\right)-E V\right)\right)\right\}\left\{e^{p \eta}\left(x^{*}, \bar{a}\right)-1\right\} \geq 0
$$

As $\sum_{i=1}^{s} \stackrel{t}{i}_{i}^{*}\left(l_{i}\left(\cdot, \bar{y}_{i}\right)+(\cdot)^{T} D w-\bar{k}\left(m_{i}\left(\cdot, \bar{y}_{i}\right)-(\cdot)^{T} E V\right)\right)$ is strictly $\mathrm{B}_{-}(\mathrm{p}, \mathrm{r})$-invex with respect to $\eta \mathrm{b}$ at $\bar{a}$ on $\operatorname{SUS}_{1}$. Then, by definition of strictly $B-(p, r)$-invexity and from above inequality, it follows that

$$
\left\{\begin{array}{c}
\frac{1}{r} b\left(x^{*}, \bar{a}\right) \times \sum_{i=1}^{s} t_{i}\left(l_{i}\left(x^{*}, \bar{y}_{i}\right)+x^{* T} D w-\bar{k}\left(m_{i}\left(x^{*}, \bar{y}_{i}\right)-x^{*} E V\right)\right) \\
\left.\quad-\sum_{i=1}^{s} t_{i}\left(l\left(\bar{a}, \bar{y}_{i}\right)+\bar{a}^{T} D w-\bar{k}\left(m_{i}(\bar{a},\} \bar{y}_{i}\right)-\bar{a}^{T} E V\right)\right)-1>0
\end{array}\right.
$$

From the hypothesis $b\left(x^{*}, \bar{a}\right)>0$, and the above inequality,

## We get

$$
\begin{aligned}
& \sum_{i=1}^{s} t_{i}\left(l_{i}\left(x^{*}, \bar{y}_{i}\right)+x^{* T} D w-\bar{k}\left(m_{i}\left(x^{*}, \bar{y}_{i}\right)-x^{* T} E V\right)\right) \\
- & \sum_{i=1}^{s} t_{i}\left(l_{i}\left(\bar{a}, \bar{y}_{i}\right)+\bar{a}^{T} D w-\bar{k}\left(m_{i}\left(\bar{a}, \bar{y}_{i}\right)-\bar{a}^{T} E V\right)\right)>0
\end{aligned}
$$

Therefore, by (13),

$$
\sum_{i=1}^{s} t_{i}\left(l_{i}\left(x^{*}, \bar{y}_{i}\right)+x^{* T} D w-\bar{k}\left(m_{i}\left(x^{*}, \bar{y}_{i}\right)-x^{* T} E V\right)\right)>0
$$

Since $t_{i} \geq 0, i=1,2, \ldots . s$, therefore there exists $i^{*}$ such that

$$
l_{i}\left(x^{*}, \bar{y}_{i}^{*}\right)+x^{* T} D w-\bar{k}\left(m_{i}\left(x^{*}, \bar{y}_{i}^{*}\right)-x^{* T} E V\right)>0
$$

Hence, we obtain the following inequality

$$
\frac{l_{i}\left(x^{*}, \bar{y}^{*}\right)+\left(x^{* T} D x^{*}\right)^{\frac{1}{2}}}{m_{i}\left(x^{*}, \bar{y}^{*}\right)-\left(x^{*}{ }^{T} E x^{*}\right)^{\frac{1}{2}}}>\bar{K}
$$

which contradicts (23). Hence the result.

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