On The Surd Equation

\[ \sqrt{2z} = \sqrt{x + iy} + \sqrt{x - iy} \]

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Abstract:

In this short paper, non-zero integer distinct integer solutions to the surd equation with three unknowns given by \[ \sqrt{2z} = \sqrt{x + iy} + \sqrt{x - iy} \] are obtained through the integer solutions of Pythagorean equation.

Keywords: surd equation, transcendental equation, integer solutions

Introduction:

Diophantine equations have an unlimited field of research by reason of their variety. Most of the Diophantine problems solved by the researchers are algebraic equations [1,2].

It seems that much work has not been done in finding the integer solutions to transcendental equations involving surds. In this context, refer [3-18] to the integral solutions of transcendental equations involving surds. This short communication analyses a transcendental equation with three unknowns given by \[ \sqrt{2z} = \sqrt{x + iy} + \sqrt{x - iy} \]. Infinitely many non-zero integer triples \((x, y, z)\) satisfying the above equation are obtained through employing the integer solutions to the well-known Pythagorean equation.
Method of analysis:

The surd equation to be solved is

\[ \sqrt{2z} = \sqrt{x + iy} + \sqrt{x - iy} \]  \hspace{1cm} (1)

On squaring both sides of (1), it simplifies to

\[ z = x + \sqrt{x^2 + y^2} \]  \hspace{1cm} (2)

To eliminate the square-root on the R.H.S. of (2), take

\[ x^2 + y^2 = \alpha^2 \]  \hspace{1cm} (3)

which is nothing but the well-known Pythagorean equation satisfied by

\[ x = r^2 - s^2, \ y = 2rs, \ r \geq s \geq 0 \]  \hspace{1cm} (4)

and

\[ \alpha = r^2 + s^2 \]

In view of (2), it is seen that

\[ z = 2r^2 \]  \hspace{1cm} (5)

Thus, (4) and (5) represent the integer solutions to (1).

A few numerical solutions are presented in Table 1 below.

<table>
<thead>
<tr>
<th>r</th>
<th>s</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>12</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>16</td>
<td>30</td>
<td>50</td>
</tr>
</tbody>
</table>

It is worth mentioning that, (3) is also satisfied by

\[ x = 2rs, \ y = r^2 - s^2, \ r \geq s \geq 0 \]  \hspace{1cm} (6)

and

\[ \alpha = r^2 + s^2 \]

From (2), the value of z is given by

\[ z = (r + s)^2 \]  \hspace{1cm} (7)

Thus, (6) and (7) satisfy (1).
A few numerical solutions are presented in Table:2 below

<table>
<thead>
<tr>
<th>r</th>
<th>s</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>4</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>12</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>16</td>
<td>12</td>
<td>36</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>30</td>
<td>16</td>
<td>64</td>
</tr>
</tbody>
</table>

Further, (3) is also satisfied by

\[
x = (r^2 + s^2) \left[ (A^2 - B^2)(r^2 - s^2) - 4rs AB \right]
\]
\[
y = (r^2 + s^2) \left[ (A^2 - B^2) 2rs + 2AB(r^2 - s^2) \right]
\]

and
\[
\alpha = (r^2 + s^2)^2 (A^2 + B^2)
\]

From (2), the value of \( z \) is given by

\[
z = 2(r^2 + s^2)(Ar - Bs)^2
\]

Thus, (8) and (9) satisfy (1). It should be remembered that the values of \( r, s, A \) and \( B \) are chosen so that \( x \) and \( y \) are non-zero positive integers as they represent the legs of a Pythagorean triangle.

A few numerical solutions are presented in Table:3 below

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>r</th>
<th>s</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>4</td>
<td>1</td>
<td>17*13</td>
<td>17*84</td>
<td>34*49</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>17*72</td>
<td>17*154</td>
<td>34*121</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>26*128</td>
<td>26*504</td>
<td>52*324</td>
</tr>
</tbody>
</table>

**Conclusion:**

In this paper, we have presented integer solutions to the surd equation with three unknowns given by

\[
\sqrt{2z} = \sqrt{x + iy} + \sqrt{x - iy}
\]

To conclude one may attempt to find integer solutions to other choices of surd equations with unknowns three or more than three.
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