



## Observations on the Pell Equation

$$x^2 = 6(y^2 + y) + 4$$

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**Abstract :** This paper concerns with the problem of obtaining non-zero distinct integer solutions to the positive pell equation represented by the binary quadratic equation  $x^2 = 6(y^2 + y) + 4$ . A few interesting relations among the solutions are presented. Further, by considering suitable linear combinations among the solutions of the considered hyperbola, the other choices of hyperbolas, parabolas,  $2^{nd}$  order Ramanujan numbers, sequence of diophantine 3-tuples with suitable property are presented.

**IndexTerms -** Positive pell equation, binary quadratic, hyperbola, parabola,  $2^{nd}$  order Ramanujan numbers, sequence of diophantine 3-tuples.

### I. INTRODUCTION

One of the areas of Number theory that has attracted many mathematicians since antiquity is the subject of diophantine equations. A diophantine equation is a polynomial equation in two or more unknowns such that only the integer solutions are determined. No doubt that diophantine equation possess supreme beauty and it is the most powerful creation of the human spirit. A pell equation is a type of non-linear diophantine equation in the form  $y^2 - Dx^2 = \pm 1$  where  $D > 0$  and square-free. The above equation is also called the Pell-Fermat equation. In Cartesian co-ordinates, this equation has the form of a hyperbola. The binary quadratic diophantine equation having the form

$$y^2 = Dx^2 + N \quad (N > 0; D > 0, \text{ a non-square integer})$$

is referred to as the positive form of the Pell equation and one may refer [1-10] for a few illustrations on Pell equations.

As quadratic Diophantine equations are rich in variety, the above references motivated us for determining integer solutions to other choices of positive Pell equation.

In this communication, the positive Pell equation  $x^2 = 6(y^2 + y) + 4$  is considered for obtaining non-zero distinct integer solutions. Further, by considering suitable linear combinations among the solutions of the considered hyperbola, the other choices of hyperbolas, parabolas,  $2^{nd}$  order Ramanujan numbers, sequence of diophantine 3-tuples with suitable property are presented.

### II. METHOD OF ANALYSIS

The hyperbola represented by the non-homogeneous quadratic equation under consideration is

$$x^2 = 6(y^2 + y) + 4 \tag{1}$$

After performing some algebra, (1) is written as

$$Y^2 = 6x^2 - 15 \tag{2}$$

where

$$Y = 6y + 3 \tag{3}$$

The smallest positive integer solution to (2) is  $x_0 = 2, Y_0 = 3$

To find the other solutions to (1), consider the corresponding pellian equation given by

$$Y^2 = 6x^2 + 1 \tag{4}$$

whose the general solution  $\tilde{x}_n, \tilde{Y}_n$  is

$$\tilde{Y}_n = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{2\sqrt{6}} g_n$$

where

$$\begin{aligned} f_n &= (5+2\sqrt{6})^{n+1} + (5-2\sqrt{6})^{n+1} \\ g_n &= (5+2\sqrt{6})^{n+1} - (5-2\sqrt{6})^{n+1}, \quad n=0,1,2,\dots \end{aligned}$$

Employing the lemma of Brahmagupta between the solutions  $(x_0, Y_0)$  &  $(\tilde{x}_n, \tilde{Y}_n)$ , the general solution  $(x_{n+1}, Y_{n+1})$  to (2) is given by

$$\begin{aligned} x_{n+1} &= x_0 \tilde{Y}_n + Y_0 \tilde{x}_n \\ &= f_n + \frac{\sqrt{6}}{4} g_n \\ Y_{n+1} &= Y_0 \tilde{Y}_n + D x_0 \tilde{x}_n \\ &= 3 * \frac{1}{2} f_n + \sqrt{6} * g_n \end{aligned} \tag{5}$$

In view of (3),

$$y_{n+1} = \frac{1}{12} (3f_n + 2\sqrt{6}g_n - 6) \tag{6}$$

Thus, (5) and (6) represented the integer solutions to (1).

A few numerical solutions to (1) are presented in Table: 1 below:

**Table: 1 Numerical solutions**

$n$	$x_{n+1}$	$y_{n+1}$
-1	2	0
0	16	6
1	158	64
2	1564	638
3	15482	6320
4	153256	62566

#### Observations:

- The values of  $x_n$  and  $y_n$  are even
- A few interesting relations among the solutions are given below:

- $x_{n+3} - 10x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 10y_{n+2} + y_{n+1} = 4$
- $12y_{n+1} + 6 = x_{n+2} - 5x_{n+1}$
- $12y_{n+2} + 6 = -x_{n+1} + 5x_{n+2}$
- $12y_{n+3} + 6 = -5x_{n+1} + 49x_{n+2}$
- $120y_{n+1} + 60 = x_{n+3} - 49x_{n+1}$
- $24y_{n+2} + 12 = x_{n+3} - x_{n+1}$
- $120y_{n+3} + 60 = 49x_{n+3} - x_{n+1}$
- $2x_{n+1} = y_{n+2} - 5y_{n+1} - 2$
- $2x_{n+2} = 5y_{n+2} - y_{n+1} + 2$
- $2x_{n+3} = 49y_{n+2} - 5y_{n+1} + 22$

- Expressions representing square integers:

- $\frac{1}{5} [13x_{2n+2} - x_{2n+3} + 10]$
- $\frac{1}{50} [129x_{2n+2} - x_{2n+4} + 100]$
- $\frac{1}{5} [4y_{2n+3} - 32y_{2n+2} - 4]$
- $\frac{1}{5} [8x_{2n+2} - 12y_{2n+2} + 4]$

➤ Expressions representing cubical integers:

- $\frac{1}{5}[13x_{3n+3} - x_{3n+4} + 3(13x_{n+1} - x_{n+2})]$
- $\frac{1}{50}[129x_{3n+3} - x_{3n+5} + 3(129x_{n+1} - x_{n+3})]$
- $\frac{1}{5}[4y_{3n+4} - 32y_{3n+3} + 12y_{n+2} - 96y_{n+1} - 56]$
- $\frac{1}{5}[8x_{3n+3} - 12y_{3n+3} + 24x_{n+1} - 36y_{n+1} - 24]$

➤ Expressions representing biquadratic integers:

- $\frac{1}{5}(13x_{4n+4} - x_{4n+5}) + \frac{4}{25}(13x_{n+1} - x_{n+2})^2 - 2$
- $\frac{1}{5}(13x_{4n+5} - x_{4n+6}) + \frac{4}{5}(13x_{2n+2} - x_{2n+3} + 10) - 2$
- $\frac{1}{50}(129x_{4n+4} - x_{4n+6}) + \frac{4}{50}(129x_{n+1} - x_{n+3})^2 - 2$
- $\frac{1}{50}(129x_{4n+4} - x_{4n+6}) + \frac{4}{50}(129x_{2n+2} - x_{2n+4} + 100) - 2$
- $\frac{1}{5}(4y_{4n+5} - 32y_{4n+4} - 14) + \frac{4}{25}(4y_{n+2} - 32y_{n+1} - 14)^2 - 2$
- $\frac{1}{5}(4y_{4n+5} - 32y_{4n+4} - 14) + \frac{4}{5}(4y_{2n+3} - 32y_{2n+2} - 4) - 2$
- $\frac{1}{5}(8x_{4n+4} - 12y_{4n+4} - 6) + \frac{4}{5}(8x_{n+1} - 12y_{n+1} - 6)^2 + 2$
- $\frac{1}{5}(8x_{4n+4} - 12y_{4n+4} - 6) + \frac{4}{5}(8x_{2n+2} - 12y_{2n+2} + 4) + 2$

➤ Employing linear combinations among the solutions, one obtains solutions to other choices of hyperbolas

**Choice1:** Let  $X = x_{n+2} - 8x_{n+1}, Y = 13x_{n+1} - x_{n+2}$

Note that  $(X, Y)$  satisfies the hyperbola

$$3Y^2 - 8X^2 = 300$$

**Choice2:** Let  $X = x_{n+3} - 79x_{n+1}, Y = 129x_{n+1} - x_{n+3}$

Note that  $(X, Y)$  satisfies the hyperbola

$$3Y^2 - 8X^2 = 48 * (25)^2$$

**Choice3:** Let  $X = 78y_{n+1} - 6y_{n+2} + 36, Y = 4y_{n+2} - 32y_{n+1} - 14$

Note that  $(X, Y)$  satisfies the hyperbola

$$6Y^2 - X^2 = 600$$

**Choice4:** Let  $X = 48y_{n+1} - 12x_{n+1} + 24, Y = 8x_{n+1} - 12y_{n+1} - 6$

Note that  $(X, Y)$  satisfies the hyperbola

$$6Y^2 - X^2 = 600$$

➤ Employing linear combinations among the solutions, one obtains solutions to other choices of parabolas

**Choice1:** Let  $X = x_{n+2} - 8x_{n+1}, Y_1 = 13x_{2n+2} - x_{2n+3} + 10$

Note that  $(X, Y_1)$  satisfies the parabola

$$15Y_1 - 8X^2 = 300$$

**Choice2:** Let  $X = x_{n+3} - 79x_{n+1}, Y_1 = 129x_{2n+2} - x_{2n+4} + 100$

Note that  $(X, Y_1)$  satisfies the parabola

$$75Y_1 - 4X^2 = 25 * 600$$

**Choice3:** Let  $X = 78y_{n+1} - 6y_{n+2} + 36, Y_1 = 4y_{2n+3} - 32y_{2n+2} - 4$

Note that  $(X, Y_1)$  satisfies the parabola

$$30Y_1 - X^2 = 600$$

**Choice4:** Let  $X = 48y_{n+1} - 12x_{n+1} + 24, Y_1 = 8x_{2n+2} - 12y_{2n+2} + 4$

Note that  $(X, Y_1)$  satisfies the parabola

$$30Y_1 - X^2 = 600$$

➤ Considering suitable values of  $x_n$  and  $y_n$ , one generates  $2^{nd}$  order Ramanujan numbers with base integers as real integers

**For illustration, consider**

$$y_2 = 64 = 1 \times 64 = 2 \times 32 = 4 \times 16 = 8 \times 8 \quad (*)$$

$$\text{Now, } 1 \times 64 = 2 \times 32$$

$$\rightarrow (1+64)^2 + (2-32)^2 = (1-64)^2 + (2+32)^2$$

$$\rightarrow 65^2 + (-30)^2 = (-63)^2 + 34^2 = 5125$$

$$1 \times 64 = 4 \times 16$$

$$\rightarrow (1+64)^2 + (4-16)^2 = (1-64)^2 + (4+16)^2 = 4369$$

$$2 \times 32 = 4 \times 16$$

$$\rightarrow (2+32)^2 + (4-16)^2 = (2-32)^2 + (4+16)^2 = 1300$$

Also,

$$2 \times 32 = 4 \times 16 \rightarrow 17^2 - 15^2 = 10^2 - 6^2$$

$$\rightarrow 17^2 + 6^2 = 15^2 + 10^2 = 325$$

Thus, 5125, 4369, 1300, 325 represent  $2^{nd}$  order Ramanujan numbers with base integers as real integers.

➤ Considering suitable values of  $x_n$  &  $y_n$ , one generates  $2^{nd}$  order Ramanujan numbers with base integers as Gaussian integers

**For illustration, consider again  $y_2$  represented by (\*)**

$$\text{Now, } 1 \times 64 = 2 \times 32 \rightarrow (1+i64)^2 + (32-i2)^2 = (1-i64)^2 + (32+i2)^2 = -3075$$

$$\text{Also, } 1 \times 64 = 2 \times 32 \rightarrow (64+il)^2 + (2-i32)^2 = (64-il)^2 + (2+i32)^2 = 3075$$

Note that -3075 & 3075 represent  $2^{nd}$  order Ramanujan numbers with base integers as gaussian integers.

In a similar manner, other  $2^{nd}$  order Ramanujan numbers are obtained

### Formation of sequence of Diophantine 3-tuples:

Consider the solution to (1) given by

$$x_1 = 16 = a \text{ (say)}, y_1 = 6 = c_0 \text{ (say)}$$

It is observed that

$$ac_0 + k^2 + 18k - 15 = (k+9)^2, \text{ a perfect square}$$

The pair  $(a, c_0)$  represents diophantine 2-tuple with property  $D(k^2 + 18k - 15)$ .

If  $c_1$  is the  $3^{rd}$  tuple, then it satisfies the system of double equations

$$16c_1 + k^2 + 18k - 15 = p^2 \quad (1^*)$$

$$6c_1 + k^2 + 18k - 15 = q^2 \quad (2^*)$$

Eliminating  $c_1$  between  $(1^*)$  and  $(2^*)$ , we have

$$10k^2 + 180k - 150 = 16q^2 - 6p^2 \quad (3^*)$$

Taking

$$p = X + 16T, q = X + 6T \quad (4^*)$$

in  $(3^*)$  and simplifying, we get

$$X^2 = 96T^2 + k^2 + 18k - 15$$

which is satisfied by

$$X = k + 9, T = 1$$

In view of  $(4^*)$  and  $(1^*)$ , it is seen that

$$c_1 = 2k + 40$$

Note that  $(16, 6, 2k+40)$  represents diophantine 3-tuple with property  $D(k^2 + 18k - 15)$ .

The process of obtaining sequences of diophantine 3-tuples with property  $D(k^2 + 18k - 15)$  is illustrated below:

Let M be a  $3 \times 3$  square matrix given by

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\text{Now, } (16, 6, 2k+40)M = (16, 2k+40, 4k+106)$$

Note that

$$16(2k+40) + (k^2 + 18k - 15) = \text{perfect square}$$

$$16(4k+106) + (k^2 + 18k - 15) = \text{perfect square}$$

$$(2k+40)(4k+106) + (k^2 + 18k - 15) = \text{perfect square}$$

Therefore, the triple  $(16, 2k+40, 4k+106)$  represents diophantine 3-tuple with property  $D(k^2 + 18k - 15)$ . The repetition of the above process leads to sequences of diophantine 3-tuples whose general form  $(a, c_{s-1}, c_s)$  is given by

$$(16, 16s^2 + (2k-14)s - 2k + 4, 16s^2 + s(2k+18) + 6), \quad s = 1, 2, 3, \dots$$

A few numerical illustrations are given in Table: 2 below:

**Table: 2 Numerical illustrations**

$k$	$(a, c_0, c_1)$	$(a, c_1, c_2)$	$(a, c_2, c_3)$	$D(k^2 + 18k - 15)$
0	(16, 6, 40)	(16, 40, 106)	(16, 106, 204)	D(-15)
1	(16, 6, 42)	(16, 42, 110)	(16, 110, 206)	D(4)
2	(16, 6, 44)	(16, 44, 114)	(16, 114, 208)	D(25)

It is noted that the triple  $(c_{s-1}, c_s + 16, c_{s+1}), s = 1, 2, 3, \dots$  forms an arithmetic progression.

In a similar way, one may generate sequences of diophantine 3-tuples with suitable property through the other solutions to (1).

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