Uniform spaces

Dr. Ranjan Kumar Singh

Abstract :-

This paper deals with the concepts in the theory of uniform spaces. We also observe that the neighbourhood system of X for each U in the uniformity and consequently the family of all sets U \( [X] \) for U in \( \nu \) is the base for the neighbourhood. Since again we conclude that the uniformity is inherited by subsets of a uniform space by restriction.

Key-words: Uniform Structure, Subbase, Uniform Space, Interior, diagonal.

Introduction :-

In the mathematical field of topology a uniform space in a set with a uniform structure. A uniform structure on a non-empty set X was first defined by A. Weil (1937) in terms of subsets of \( X \times X \). J.W. Tukey (1940) later provided as alternative description of a uniform structure using covers of X.

Basic concepts in the theory of uniform spaces:

Let \( X \) be a non-empty set. For arbitrary subsets \( U \) and \( V \) of \( X \times X \), we write \( V^{-1} = \{(y,x) : (x,y) \in V\} \) and \( U \circ V = \{(x,y) : \exists z \in X \text{ such that } (x,z) \in V \text{ and } (z,y) \in U\} \). It follows easily that \( U \circ (V \circ W) = (U \circ V) \circ W \) and \( (U \circ V)^{-1} = V^{-1} \circ U^{-1} \). We shall write \( U^2 \) for \( U \circ U \). The diagonal of \( X \times X \) which is denoted by \( \Delta (x) \) or simply \( \Delta \) defined as the set \( \{(x,x) : x \in X\} \). For each subsets \( A \) of \( X \) the set \( U[A] \) is defined to be \( \{y : (x,y) \in U \text{ for some } x \in A\} \). We write \( U[x] \) or \( U[\{x\}] \) if \( x \) is a point in \( X \). For each \( U \) and \( V \) and each \( A \) it is true that \( (U \circ V)[A] = U[V[A]] \).

Clearly \( (U^{-1})^{-1} = U \), \( U \) is said to be symmetric if \( U^{-1} = U \).

Definition:

A uniformity or uniform structure for a set \( X \) is a non-empty family \( \mathcal{U} \) of subsets of \( X \times X \) which satisfy the following conditions:

(i) Each member of \( \mathcal{U} \) contains the diagonal \( \Delta \);

(ii) if \( U \in \mathcal{U} \), then \( U^{-1} \in \mathcal{U} ; \)

(iii)If \( U \in \mathcal{U} \), then \( \exists V \in \mathcal{U} \text{ such that } V^2 \subseteq U \);

(iv)If \( U \in \mathcal{U} \text{ and } U \subseteq V \subseteq X \times X \), then \( V \in \mathcal{U} \text{ and } \)
(v) If \( U \) and \( V \) are members of \( \mathcal{U} \), then \( \bigcup \cap V \in \mathcal{U} \); Elements of \( \mathcal{U} \) are said to be vicinities. A uniform space is a set together with a uniformity for it. Thus the pair \((X, \mathcal{U})\) is a uniform space.

**Definition:**

(i) A subfamily \( \mathcal{B} \) for a uniformity \( \mathcal{U} \) is a base for \( \mathcal{U} \), iff each member of \( \mathcal{U} \) contains a member of \( \mathcal{B} \).

(ii) If \( \mathcal{B} \) is a base for \( \mathcal{U} \); then \( \mathcal{B} \) determines \( \mathcal{U} \) entirely, for a subsets \( U \) of \( X \times X \) belongs to \( \mathcal{U} \) if \( U \) contains a member of \( \mathcal{B} \).

**Definition:**

(i) A subfamily \( \mathcal{B} \) is a subbase for \( \mathcal{U} \) if the family of finite intersections

(ii) of members of \( \mathcal{B} \) is a base for \( \mathcal{U} \).

(iii) We now state the following theorem, the proof of which is simple.

**Theorem:**

A non-empty family \( \mathcal{B} \) of subsets of \( X \times X \) is a base for some uniformity for \( X \) if and only if

(i) Each member of \( \mathcal{B} \) contains the diagonal \( \Delta \);

(ii) If \( U \in \mathcal{B} \), then \( \exists V \in \mathcal{B} \) such that \( V \subseteq U^{-1} \);

(iii) If \( U \in \mathcal{B} \), then \( \exists V \in \mathcal{B} \) such that \( V^2 \subseteq U \);

(iv) If \( U, V \in \mathcal{B} \) then \( \exists W \in \mathcal{B} \) such that \( W \subseteq U \cap V \).

**Proof:**

We have to show that the family \( \mathcal{B} \) of finite intersections of member of \( \mathcal{B} \) satisfies the condition of theorem (5.1).

If \( U_1, U_2, \ldots, U_n \) and \( V_1, V_2, \ldots, V_n \) are subsets of \( X \times X \) all belonging to \( \mathcal{B} \) and if \( U = \bigcap_{i=1}^{n} U_i \) and

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V = \bigcap_{i=1}^{n} V_i \text{ then } V \subseteq U^{-1} \left( \text{or } V^2 \subseteq U \right) \text{ whenever } V_i \subseteq U_i^{-1} \text{ (respectively, } V^2_i \subseteq U_i \text{) for each } i. \text{ From this observation the proof of this theorem follows.}

**Definition:**

If \((X, \mathcal{U})\) is a uniform space the topology \( J \) of the uniformity \( \mathcal{U} \), or the uniform topology is the family of all subsets \( T \) of \( X \) such that for each \( x \in T \) there is \( U \in \mathcal{U} \) such that \( U \left[ x \right] \subseteq T \).
To verify that $J$ is a topology is simple. In fact the union of members of $J$ is surely a member of $J$. If $T$ and $S$ are members of $J$ and $x \in T \cap S$, there are $U$ and $V$ in $\mathcal{U}$ such that $U[x] \subseteq T$ and $V[x] \subseteq S$, and hence $U \cap V[x] \subseteq T \cap S$ consequently $T \cap S \in J$ and $J$ is a topology.

**Theorem:**
The interior of a subset $A$ of $X$ relative to the uniform topology is the set of all points $x$ such that $U[x] \subseteq A$ for some $U$ in $\mathcal{U}$.

**Proof:**
To prove the theorem it is sufficient to prove that the set $B = \{ X : U[X] \subseteq A \text{ for some } U \in \mathcal{U} \}$ is open relative to the uniform topology, for $B$ surely contains every open subset of $A$ and, if $B$ is open, then $\exists U \in \mathcal{U}$ such that $U[X] \subseteq A$ and again $\exists V \in \mathcal{U}$ such that $V^2 \subseteq U$. If $y \in V[X]$ then $V[y] \subseteq V^2[X] \subseteq U[X] \subseteq A$ and $y \in B$. hence $V[X] \subseteq B$ and $B$ is open.

This completes the proof.

**Remark:**
It follows immediately that $U[X]_x$ is a neighbourhood system of $x$ for each $U$ in the uniformity $\mathcal{U}$, and consequently the family of all sets $U[X]$ for $U$ in $\mathcal{U}$ is a base for the neighbourhood system of $x$ (the family is actually identical with the neighbourhood system). The following theorem is then clear.

**Theorem:**
If $\mathcal{B}$ is a base (or subbase) for the uniformity $\mathcal{U}$, then for each $x$ the family of sets $U[X]$ for $U$ in $\mathcal{B}$ is a base (subbase respectively) for the neighbourhood system of $x$.

**Remark (2):**
A uniformity is inherited by subsets of a uniform space by restriction.

If $X$ is a uniform space for a uniformity $\mathcal{U}$, and $Y$ is a subset of $X$, then $Y$ is a uniform space (called subspace) under the induced (relative) uniformity $Y_u = \{ YxY \cup U : U \in \mathcal{U} \}$ for $Y$.

If $\mathcal{B}$ is a base for $\mathcal{U}$, then $Y_\mathcal{B} = \{ YxY \cup U : U \in \mathcal{B} \}$ is a base for $Y$. It can be verified that the topology of the relative uniformity $\mathcal{S}$ is the relativized topology for.

**Conclusion:** Hence, the interior of a subset $A$ of $X$ relative to the uniform topology is the set of all points $x$ such that $U[x] \subseteq A$ for some $U$ in $\mathcal{U}$ and also a uniformity is inherited by subsets of uniform space by restriction.

**Reference:**


