Study Of Serial Channels With Feedback Connected With Non-Serial Queuing Processes With Reneging And Balking With Finite Waiting Space

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Abstract

O'Brien (1954), Jackson (1954) and Hunt (1955) studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that the unit must go through each service channel without leaving the system. Barrer (1955) obtained the steady – state solution of a single channel queuing model having Poisson input, exponential holding time, random selection where impatient customers leave the service facility after a wait of certain time. Finch (1959) studied simple queues with customers at random for service at a number of service stations in series where the arrival from outside was considered at the initial stage. Feedback is permitted either from the terminal server or from each server of the series to the queue waiting for service at that stage by imposing an upper limit on the number of customers in the system at any time. Singh(1984) studied the problem of serial queues introducing the concept of reneging. Singh and Umed (1994) worked on the network of queuing processes with impatient customers. Punam (2011) found the steady-state solution of serial queuing processes where feedback is not permitted.

In our present work, the steady-state solutions are obtained for serial channels with feedback connected with non-serial queuing processes with reneging and balking and non-serial queuing processes with reneging and balking in which

(i) M service channels in series are linked with N non-serial channels having reneging and balking.
(ii) Feedback is permitted in M serial service channel from any channel to its pervious channel.
(iii) A customer may join any channel from outside and leave the system at any stage after getting service.
(iv) Poisson arrivals and exponential service times are followed.
(v) The queue disciple is random selection for service.
(vi) Waiting space is finite.

Key Words: Steady-State, difference-differential, waiting space, random selection, Poisson arrivals, exponential service, feedback, reneging and balking.

1. Formulation of Model

The system consists of the queues $Q_j\,(j=1,2,3,....,M)$ and non-serial channel $Q_{1i}\,(i=1,2,3,.......N)$ with respective servers $S_j\,(j=1,2,3,....,M)$ and $S_{1i}\,(i=1,2,3,.......N)$. Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j\,(j=1,2,......,M)$ and $\lambda_{1i}\,(i=1,2,......,N)$ at $Q_j\,(j=1,2,3,....,M)$ and $Q_{1i}\,(i=1,2,3,.......N)$, but the sight of a long queue at $Q_{1i}\,(i=1,2,3,.......N)$ may discourage the fresh customer joining it and may decide not to enter the service channel $Q_{1i}\,(i=1,2,3,.......N)$. Then the Possion input rate λ_{1i} would be $\frac{\lambda_{1i}}{m_i+1}$ where m_i is the queue size of Q_{1i}. Further the impatient customer after joining any service channel Q_{1i} may leave the queue without getting service after a wait of certain time. Service time distribution for servers $S_j\,(j=1,2,3,....,M)$ and $S_{1i}\,(i=1,2,3,.......N)$ are mutually independent, negative exponential distribution with $\mu_j\,(j=1,2,3,....,M)$ and $\mu_{1i}\,(i=1,2,3,.......N)$ respectively. After the completion of service at S_j, the customer either leave the system with probability p_j or join the next channel with probability q_j or join back the previous channel with probability r_j such that $p_j+q_j+r_j=1\,(j=1,2,3,...M-1)$ or join any of queue $Q_{1i}\,(i=1,2,3,.......N)$ with probability $\frac{q_{Mi}}{m_i+1}(i=1,2,3,....N)$ such that $p_M+r_M+\sum_{i=1}^{N}\frac{q_{Mi}}{m_i+1}=1$. It is being mentioned here that $r_j=0$ when $j=1$ as there is no previous channel of the first channel.

2. Formulation of Equations:

Define: $P(n_1,n_2,n_3,.....n_{M-1},n_M;m_1,m_2,m_3,...m_{N-1},m_N;t)$ = the probability that at time 't' there are n_j customers (which may leave the system after service or join the next phase or join back the previous channel) waiting before $S_j\left(\hat{j}=1,2,3,....,M-1,M\right)$; m_i customers (which may balk or renege) waiting respectively before the server $S_{1i}\left(i=1,2,3,.......N\right)$.

We define the operators $T_i\cdot$, $T\cdot_i$, $T\cdot_{i,i+1}\cdot, T_{i-1}\cdot,\cdot_i$ to act upon the vector $\tilde{n}=(n_1,n_2,n_3,.....n_M)$ as follows

$$T_i\cdot(\tilde{n})=(n_1,n_2,n_3,.....,n_i-1,....n_M)$$
$$T\cdot_i(\tilde{n})=(n_1,n_2,n_3,.....,n_i+1,....,n_M)$$
$$T\cdot_i,_{i+1}\cdot(\tilde{n})=(n_1,n_2,n_3,.....,n_i+1,n_{i+1}-1,....,n_M)$$
$$T_{i-1}\cdot,\cdot_i(\tilde{n})=(n_1,n_2,n_3,.....,n_{i-1}-1,n_i+1....,n_M)$$

Following the procedure given by Kelly (1979), we write the difference – differential equations as

$$\begin{aligned}\frac{d}{dt}P(\tilde{n},\tilde{m};t)=&-\left[\sum_{i=1}^{M}\lambda_i+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{i=1}^{M}\delta(n_i)\mu_i+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+C_{jm_j}\right)\right]P(\tilde{n},\tilde{m};t)\\&+\sum_{i=1}^{M}\lambda_iP(T_i\cdot(\tilde{n}),\tilde{m};t)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_j\cdot(\tilde{m});t)+\sum_{i=1}^{M}p_i\mu_iP(T\cdot_i(\tilde{n}),\tilde{m};t)\\&+\sum_{i=1}^{M-1}q_i\mu_iP(T\cdot_{i,i+1}\cdot(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M}r_i\mu_iP(T_{i-1}\cdot,\cdot_i(\tilde{n}),\tilde{m};t)\\&+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P(n_1,n_2,n_3,....n_M+1,T_j\cdot(\tilde{m});t)\\&+\sum_{j=1}^{N}\left(\mu_{1j}+C_{jm_j+1}\right)P(\tilde{n},T\cdot_j(\tilde{m});t)\end{aligned}\qquad(2.1)$$

Where

$n_i\geq 0\quad(i=1,2,3,....,M),m_j\geq 0\quad(j=1,2,3,....,N)$ and $\delta(x)=\begin{cases}1 & if\quad x\neq 0\\0 & if\quad x=0\end{cases}$

and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j<K$

and

$$\begin{aligned}\frac{d}{dt}P(\tilde{n},\tilde{m};t)=&-\left[\sum_{i=1}^{M}\delta(n_i)\mu_i+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+C_{jm_j}\right)\right]P(\tilde{n},\tilde{m};t)\\&+\sum_{i=1}^{M}\lambda_iP(T_i\cdot(\tilde{n}),\tilde{m};t)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_j\cdot(\tilde{m});t)\\&+\sum_{i=1}^{M-1}q_i\mu_iP(T\cdot_{i,i+1}\cdot(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M}r_i\mu_iP(T_{i-1}\cdot,\cdot_i(\tilde{n}),\tilde{m};t)\\&+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P(n_1,n_2,n_3,....n_M+1,T_j\cdot(\tilde{m});t)\end{aligned}\qquad(2.2)$$

Where $n_i\geq 0\quad(i=1,2,3,....,M),m_j\geq 0\quad(j=1,2,3,....,N)$ and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j=K$.

$P(\tilde{n},\tilde{m};t)=\tilde{0}$ if any of the arguments in negative.

3. Steady –State Equations:-

We write the following Steady–state equations of the queuing model by equating the time-derivates to zero in the equation (2.1)

$$\left[\sum_{i=1}^{M}\lambda_i+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{i=1}^{M}\delta(n_i)\mu_i+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+c_{jm_j}\right)\right]P(\tilde{n},\tilde{m})=\sum_{i=1}^{M}\lambda_iP(T_i.(\tilde{n}),\tilde{m})$$
$$+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_j.(\tilde{m}))+\sum_{i=1}^{M}p_i\mu_iP(T._i(\tilde{n}),\tilde{m})$$
$$+\sum_{i=1}^{M-1}q_i\mu_iP(T._{i,i+1}.(\tilde{n}),\tilde{m})+\sum_{i=1}^{M}r_i\mu_iP(T_{i-1}.,._i(\tilde{n}),\tilde{m})$$
$$+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P(n_1,n_2,n_3,....n_M+1,T_j.(\tilde{m}))$$
$$+\sum_{j=1}^{N}\left(\mu_{1j}+C_{jm_j+1}\right)P(\tilde{n},T._j(\tilde{m})) \qquad (3.1)$$

For $n_i\geq 0\quad (i=1,2,3,....,M), m_j\geq 0\ \ (j=1,2,3,....,N)$ and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j<K.$

$$\left[\sum_{i=1}^{M}\delta(n_i)\mu_i+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+C_{jm_j}\right)\right]P(\tilde{n},\tilde{m};)=$$
$$\sum_{i=1}^{M}\lambda_iP(T_i.(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_j.(\tilde{m}))$$
$$+\sum_{i=1}^{M-1}q_i\mu_iP(T._{i,i+1}.(\tilde{n}),\tilde{m})+\sum_{i=1}^{M}r_i\mu_iP(T_{i-1}.,._i(\tilde{n}),\tilde{m}) \qquad (3.2)$$
$$+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P(n_1,n_2,n_3,....n_M+1,T_j.(\tilde{m}))$$

4. Steady-State Solutions:-

The solutions of the Steady-State equations (3.1) can be verified to be

$$P(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left(\frac{\lambda_1+r_2\mu_2\rho_2}{\mu_1}\right)^{n_1}\left(\frac{\lambda_2+\mu_1q_1\rho_1+\mu_3r_3\rho_3}{\mu_2}\right)^{n_2}\left(\frac{\lambda_3+\mu_2q_2\rho_2+\mu_4r_4\rho_4}{\mu_3}\right)^{n_3}......$$
$$....\left(\frac{\lambda_{M-1}+\mu_{M-2}q_{M-2}\rho_{M-2}+\mu_Mr_M\rho_M}{\mu_{M-1}}\right)^{n_{M-1}}\left(\frac{\lambda_M+\mu_{M-1}q_{M-1}\rho_{M-1}}{\mu_M}\right)^{n_M}\frac{\left(\lambda_{11}+\mu_Mq_{M1}\rho_M\right)^{m_1}}{m_1!\prod_{j=1}^{m_1}\left(\mu_{11}+C_{1j}\right)}.$$
$$.\frac{\left(\lambda_{12}+\mu_Mq_{M2}\rho_M\right)^{m_2}}{m_2!\prod_{j=1}^{m_2}\left(\mu_{12}+C_{2j}\right)}.....\frac{\left(\lambda_{1N}+\mu_Mq_{MN}\rho_M\right)^{m_N}}{m_N!\prod_{j=1}^{m_N}\left(\mu_{1N}+C_{Nj}\right)} \qquad (4.1)$$

For $n_i\geq 0\quad (i=1,2,3,....,M), m_j\geq 0\ \ (j=1,2,3,....,N)$

Where

$$\rho_1=\left(\frac{\lambda_1+r_2\mu_2\rho_2}{\mu_1}\right)$$
$$\rho_2=\left(\frac{\lambda_2+\mu_1q_1\rho_1+\mu_3r_3\rho_3}{\mu_2}\right)$$
$$\rho_3=\left(\frac{\lambda_3+\mu_2q_2\rho_2+\mu_4r_4\rho_4}{\mu_3}\right) \qquad (4.2)$$

....
....
....

$$\rho_{M-1}=\left(\frac{\lambda_{M-1}+\mu_{M-2}q_{M-2}\rho_{M-2}+\mu_Mr_M\rho_M}{\mu_{M-1}}\right)$$
$$\rho_M=\left(\frac{\lambda_M+\mu_{M-1}q_{M-1}\rho_{M-1}}{\mu_M}\right)$$

Solving these (4.2) M-equations for ρ_M with the help of determinants, we get

$$\rho_M = \frac{\begin{array}{l}(\lambda_M \Delta_{M-1} + q_{M-1}\lambda_{M-1}\Delta_{M-2} + q_{M-1}q_{M-2}\lambda_{M-2}\Delta_{M-3} + \dots + q_{M-1}q_{M-2}\dots q_3\lambda_3\Delta_2 \\ + q_{M-1}q_{M-2}\dots q_3 q_2\lambda_2\Delta_1 + q_{M-1}q_{M-2}\dots q_3 q_2 q_2 q_1\lambda_1)\end{array}}{\mu_M\left[\Delta_{M-1} - q_{M-1}r_M\Delta_{M-2}\right]} \qquad (4.3)$$

where $\Delta_M = \Delta_{M-1} - q_{n-1}r_M\Delta_{M-2}$

$$\Delta_{M-1} = \Delta_{M-2} - q_{M-2}r_{M-1}\Delta_{M-3} \qquad (4.4)$$

Continuing in this way

$\Delta_3 = (\Delta_2 - q_2 r_3 \Delta_1)$

Where

$$\Delta_M = \begin{vmatrix} 1 & -r_2 & 0 & 0 & - & - & - & - & - & 0 & 0 & 0 \\ -q_1 & 1 & -r_3 & 0 & - & - & - & - & - & 0 & 0 & 0 \\ 0 & -q_2 & 1 & -r_4 & - & - & - & - & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & - & - & - & -q_{M-2} & 1 & -r_{M-1} \\ 0 & 0 & 0 & 0 & - & - & - & - & - & 0 & -q_{M-1} & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & -r_2 \\ -q_1 & 1 \end{vmatrix} = 1 - q_1 r_2$$

$$\Delta_1 = |1| = 1$$

Since ρ_M is obtained, we can get ρ_{M-1} by putting the value of ρ_M in the last equation of (4.2), ρ_{M-2} by putting the values of ρ_{M-1} and ρ_M in the last but one equation of (4.2), Continuing in this way, we shall obtain $\rho_{M-3}, \rho_{M-4}, ---, \rho_3, \rho_2$, and ρ_1,.

Thus, we write (4.1) as under

$$P(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})(\rho_1)^{n_1}(\rho_2)^{n_2}(\rho_3)^{n_3} ---(\rho_M)^{n_M} \cdot \frac{(\lambda_{11}+\mu_M q_{M1}\rho_M)^{m_1}}{m_1!\prod_{j=1}^{m_1}(\mu_{11}+C_{1j})} \cdot \frac{(\lambda_{12}+\mu_M q_{M2}\rho_M)^{m_2}}{m_2!\prod_{j=1}^{m_2}(\mu_{12}+C_{2j})} \cdots \frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M)^{m_N}}{m_N!\prod_{j=1}^{m_N}(\mu_{1N}+C_{Nj})}; \qquad (4.5)$$

For $n_i \geq 0 \quad (i = 1,2,3,\dots,M), m_j \geq 0 \quad (j = 1,2,3,\dots,N)$

We obtain $P(\tilde{0},\tilde{0})$ from the normalizing conditions.

$\sum_{\substack{\tilde{n}=\tilde{0}\\ \tilde{m}=\tilde{0}}}^{\infty} P(\tilde{n},\tilde{m}) = 1$ and with the restriction that traffic intensity of each service channel of the system is less than unity,

Thus $P(\tilde{n},\tilde{m})$ is completely determined.

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