Domination Number Of Dominating Graphs

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Abstract: Let G be a graph with minimal vertex dominating sets $G_1, G_2, \ldots, G_m$. Form a graph D(G) with vertices corresponding to $G_1, G_2, \ldots, G_m$ and two sets G_i and G_j are adjacent if they have atleast one vertex in common. This graph D(G) is known as Dominating Graph. Minimum cardinality of a minimal dominating sets of D(G) is called domination number of D(G) and is denoted by $\gamma(D(G))$. In this paper, bounds on $\gamma(D(G))$ are obtained and its exact values for some standard graphs are found.

Keywords: Path, Complete graph, Complete bipartite graph, Domination number, γ- set.

1. Introduction

Graphs are used as a powerful tool for representing and analyzing relationship between objects. Graphs contains vertices (also known as nodes) and edges that connect pairs of vertices. Graph theory is an inevitable area of research. They are useful in fields such as computer science, operations research , social science, biology and many more, because of its ability to depict and examine relationships. Graph theory has undergone rapid advancement over the past six decades. Among the diverse array of concepts within this field, one area that has become particularly prominent is domination in graph theory. Domination in graphs has been extensively studied and is crucial due to its relevance in wireless sensor networks, social networks, decision-making processes, and its interconnectedness with other graph theoretical concepts. Some of the applications of domination are listed in [1]. Claude Berge [2] introduced the concept of the coefficient of external stability in his book on graph theory, which is now recognized as the domination number of a graph. In 1962, Oystein Ore [4] introduced the terms dominating set and domination number in his own book on graph theory. Ore and Berge's [2,4] research on graph domination and related subjects has played a crucial role in advancing domination theory. Building on this foundation, this article introduces the concept of the domination number of dominating graphs using minimal dominating sets. Furthermore, it investigates the domination number of dominating graphs across various classes of graphs.

The article is structured as follows:

The initial section provides an overview of the fundamentals of graph theory. The concept of domination number of dominating graphs formed from various graphs like paths, complete graphs, complete bipartite graphs and star is discussed.

2. Preliminaries

All graphs considered here are finite, simple and undirected. Any undefined term in this paper may be found in Haynes et al , Harary[3, 5].

A linear sequence of distinct vertices arranged in such a manner that two consecutive vertices are adjacent is called a Path [3], denoted by P_n.

A graph G is Connected [3] if there is a path between every pair of vertices. The length of a path or a cycle is the number of its edges.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a Dominating Set [3] if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

The Domination Number $\gamma(G)$ of a graph G [5] equals the minimum cardinality of a dominating set in G.

We define a set of vertices a γ - set if it is a dominating set with cardinality $\gamma(G)$ [5].

Unless otherwise stated, a graph has n vertices and m edges.

3. Domination number of dominating graphs

Definition 3.1. Let G be a graph with minimal vertex dominating sets $G_1, G_2, ...,G_m$. Form a graph D(G) with vertices corresponding to $G_1, G_2, ...,G_m$ and two sets G_i and G_j are adjacent if they have atleast one vertex in common. This graph D(G) is known as Dominating Graph. Minimum cardinality of a minimal dominating sets of D(G) is called domination number of D(G) and is denoted by $\gamma(D(G))$.

3.1 Complete Graphs

Theorem 3.1. For Complete graph K_n, $\gamma(D(K_n)) = n$.

Let K_n be a complete graph on n vertices $v_1, v_2, ..., v_n$. Then $\{v_1\}, \{v_2\}, \{v_3\},.........,\{v_n\}$ are minimal dominating sets of K_n. So $\{v_1\}, \{v_2\}, \{v_3\},........,\{v_n\}$ represents vertices of $D(K_n)$ and also they are completely independent. i.e., $D(K_n)$ is a trivial graph on n vertices. Hence $D(K_n)$ has only one minimal dominating set $\{\{v_1\}, \{v_2\}, \{v_3\},....,\{v_n\}\}$ .Then $\gamma(D(K_n)) = n$.
Also(D(K_n) is isomorphic to K_n.

3.2 Path

Theorem 3.2. For a path P_n, $\gamma(D(P_n)) = \begin{cases} 2 & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$

Proof. Let $P_n = v_1v_2........v_n$ be a path of length $n-1$.

Case 1: If n is odd

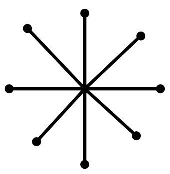
Let $S_1 = \{v_1, v_3, ..., v_n\}$ and $S_2 = \{v_2, v_4, ..., v_{n-1}\}$ are two disjoint minimal dominating sets . Then these two sets represents two non- adjacent vertices in D(G). Any other minimal dominating sets of G are adjacent to either S_1 or S_2 or both. So $\gamma(D(P_n)) = 2$. Similarly by taking $S_1 = \{v_1, v_3, ..., v_{n-1}\}$ and $S_2 = \{v_2, v_4, ..., v_n\}$, if n is even we get $\gamma(D(P_n)) = 2$.

If n = 1, $D(P_n)) = K_1$. So $\gamma(D(P_n)) = 1$.

Remark 3.1. $D(P_n)$ is a hamiltonian graph if $n \geq 4$.

3.3 Star Graphs

Theorem 3.3. For a star graph S_n, $\gamma(D(S_n)) = 2$.



S_n

Proof: In a star S_n, $n-1$ vertices $v_1, v_2, ..., v_{n-1}$ are pendant vertices and v is a vertex of degree $n-1$. Hence for them $\{v\}$ and $\{v_1, v_2, ., v_{n-1}\}$ are the minimal dominating sets. These two sets represents two non adjacent vertices in $D(S_n)$.
Hence $\gamma(D(S_n)) = 2$.

3.4 Isolated Vertex

A vertex in a graph G is an isolated vertex if it is of degree 0.

Theorem 3.4. $\gamma(D(G)) = 1$ if and only if G contains atleast one isolated vertex. In this case D(G) is complete.

Proof. Let G be a graph with atleast one isolated vertex. Then any minimal dominating sets of G contains isolated vertices. So in D(G) all vertices are adjacent to each other. Hence D(G) is a complete graph K_m where m is the number of minimal dominating sets of G.i.e., $\gamma(D(G)) = 1$.

Conversly assume that $\gamma(D(G)) = 1$. That is D(G) contains a vertex of degree $m - 1$, where $m = |V(D(G))|$ is the number of minimal dominating sets in G.
Suppose G has no isolated vertex. Then for any minimal dominating set S, $V - S$ is a dominating set of G. So G has another minimal dominatiing set $S_1 \subseteq V - S$. But S and S_1 are independent. So these two vertices in D(G) are nonadjacent. This is true for any minimal dominating set S of G. Hence domination number of D(G) is not equal to one, which is a contradiction. i.e.,
$\gamma(D(G)) = 1$ if and only if G contains atleast one isolated vertex.

Theorem 3.5. If a graph G contains a vertex of degree $n - 1$, then D(G) has atleast one isolated vertex.

Proof. If G contains a vertex v of degree $(n - 1)$, then $\{v\}$ is a minimal dominating sets of G. So the vertex corresponding to $\{v\}$ is an isolated vertex in D(G).

Remark 3.2. If G is a trivial graph on $n \geq 2$ vertices $v_1, v_2, ...v_n$, then $\{v_1, v_2, ...,v_n\}$ is the only one minimal dominating set in G. So D(G) is K_1.
i.e, D(G) has one isolated vertex. But G has no vertex of degree $n - 1$. So the converse is not true in general.

If the collection of all minimal dominating sets of a graph G forms a partition of V (G), then $\gamma(D(G))$ = number of minimal dominating sets of G.
Otherwise $\gamma(D(G))$ < number of minimal dominating sets of G.

3.5 Complete Bipartite Graphs

Theorem 3.6. Let $G = K_{p,q}$ where $V = V_1 \cup V_2$ is the bi-partition with $|V_1| = p, |V_2| = q$. Then $\gamma(D(G)) = 2$, if $p, q > 1$.

Proof. Let $K_{p,q}$ be a complete bipartite graph with bipartition V_1, V_2. Then set of all vertices of V_1, set of all vertices of V_2 are two independent minimal dominating sets of $K_{p,q}$. These two corresponding vertices in $D(K_{p,q})$ are adjacent to any vertex in $D(K_{p,q})$.
Hence $\gamma(D(K_{p,q})) = 2$.

4 Conclusion

In this paper, we described dominating graphs and minimal domination number of dominating graphs of some standard graphs are studied. We chracterize the graph with domination number of dominating graph is 1. Further research on this topic will include the domination number of dominating graph of product graphs and application of this concept.

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