NUMERICAL ACCURACY OF SIMPSON'S THREE-EIGHTH RULE

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Abstract: Five definite integrals have been solved by Simpson's three-eighth rule at different values of upper limits. The values of integrals have been compared to the exact values to get the error in the value of integrals. Highest value of error in each integral possesses at least six consecutive zeros after decimal point which indicates that the Simpson's three-eighth rule is accurate up to 6 places after the decimal point.

Key words: Definite integral, Simpson's three-eighth rule, numerical accuracy, error, Simpson's one-third rule.

Introduction

Popular methods use one of the Newton–Cotes formulas (like the midpoint rule or Simpson's rule) or Gaussian quadrature [1-5]. These methods rely on a "divide and conquer" strategy, whereby an integral on a relatively large set is broken down into integrals on smaller sets. In higher dimensions, where these methods become prohibitively expensive in terms of computational effort, one may use Monte Carlo or quasi-Monte Carlo methods or, in modestly large dimensions, the method of sparse grids.

More accurate [6-9] integration formulas with smaller truncation error can be obtained by interpolating several data points with higher-order interpolating polynomials. For example, the fourth-order interpolating polynomial $P_4(t)$ between five data points leads to the Boole's rule of numerical integration. The Boole's rule has the global truncation error of order $O(h^6)$. However, the higher-order interpolating polynomials often do not provide good approximations for integrals because they tend to oscillate wildly between the samples (polynomial wiggle). As a result, they are seldom used past Boole's rule. Another popular numerical algorithm is used instead to reduce the truncation error of numerical integration. This is Romberg integration based on the Richardson extrapolation algorithm

In numerical analysis, the Newton–Cotes formulas are a group of formulas for numerical integration (also called quadrature) based on evaluating the integrand at n+1 equally-spaced points. Newton–Cotes formulas can be useful if the value of the integrand at equally-spaced points is given. If it is possible to change the points at which the integrand is evaluated, then other methods such as Gaussian quadrature [9-10] and Clenshaw–Curtis quadrature [11-12] are probably more suitable.

Material and Method

In order to find out the relative numerical accuracy of quadrature formulas, definite integrals given in Table-1 have been calculated the by dividing the interval of integration [a, b] into 288 equal parts using Simpson's three-eighth rule [13-17] with the help of computer program developed by us in C++ language. Simpson's three-eighth rule is as under-

$$\int_{x_1}^{x_n} f(x)dx = \sum_{k=1}^{(n-1)/3} \int_{x_{3k-2}}^{x_{3k+1}} f(x)dx$$

$$= \sum_{k=1}^{(n-1)/3} h[3f(x_{3k-2}) + 9f(x_{3k-1}) + 9f(x_{3k}) + 3f(x_{3k+1})]/8$$

where $h = x_{3k+1} - x_{3k}$, $(n-1)/3$ are positive integers.

Table-1: Definite integrals and their exact values

S. No.	Definite Integral	Exact Value
1	$\int_0^1 (12x^5 + 5x^4 - 6x^2 + 4)\, dx$	5.000000
2	$\int_1^3 \left(\frac{2}{x^2} + 3x\right) dx$	13.333333
3	$\int_0^1 2^x\, dx$	1.442695
4	$\int_{-1}^1 e^{2x-1}\, dx$	1.334247
5	$\int_0^2 \frac{3x}{x^2+2}\, dx$	1.647918

Result and Discussion

Calculation of values of integral-1 at different upper limits using Simpson's three-eighth rule:-

Values of integral have been calculated at different values of upper limit by using Simpson's three-eighth rule with help of computer program developed in C++ programming language. Calculated values by using Simpson's three-eighth rule at different values of upper limit, exact values and error in calculated values is included in Table-2. Graph of errors in the calculated values by Simpson's three-eighth rule at different upper limit is given in Graph-1. Statistics of error is as under-

Lowest value of error	=	0.000000000928796
Highest value of error	=	0.000000399849530
Average error	=	0.000000059124917
Lowest percentage error	=	0.000000634481747
Highest percentage error	=	0.000008248767552
Average percentage error	=	0.000002539577735

It is evident from Table-2 that the numerical accuracy of Simpson's three-eighth rule is up to 6 places after the decimal point.

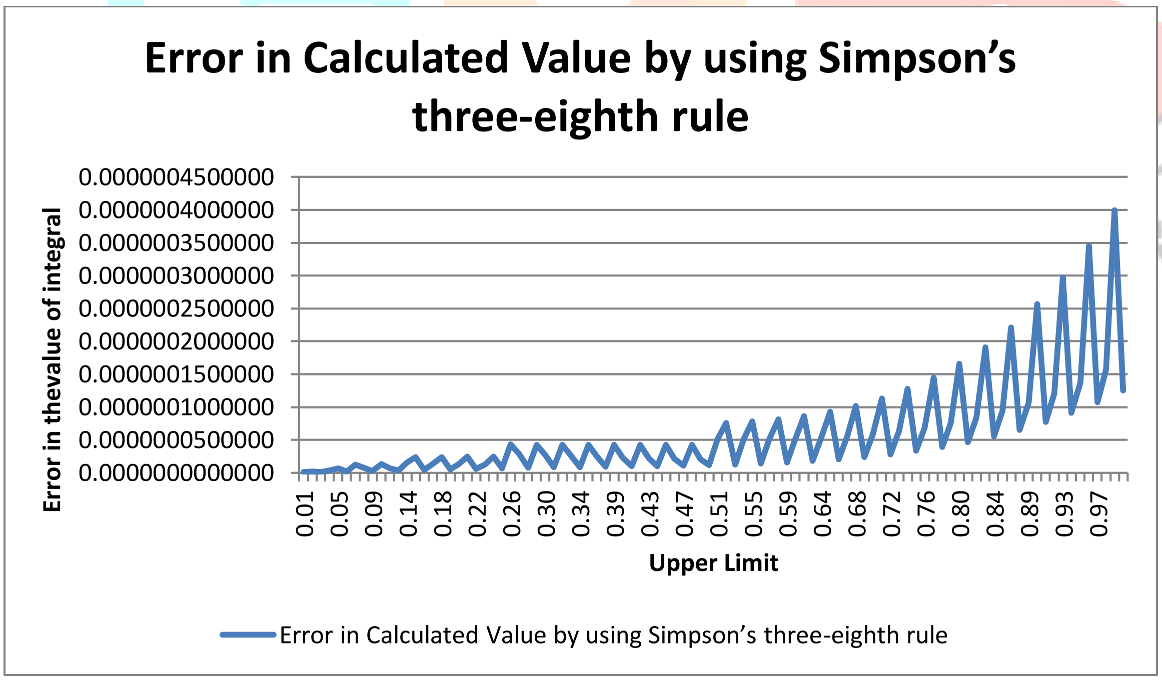
Table-2: Calculated values of integral-1 by using Simpson's three-eighth rule at different values of upper limit and constant value 0 of lower limit, exact values and error in calculated values

S. No.	Upper Limit	Value Calculated By Simpson's Three-Eighth Rule	Exact Value	Error in Calculated Value by using Simpson's three-eighth rule
1	0.010	0.041664406543288	0.041664407472084	0.000000000928796
2	0.021	0.083315253555495	0.083315255412613	0.000000001857118
3	0.031	0.124938997445670	0.124938996508718	0.000000000936952
4	0.042	0.166522128040324	0.166522131747183	0.000000003706859
5	0.052	0.208051187909089	0.208051181401884	0.000000006507205
6	0.062	0.249512793500909	0.249512791633606	0.000000001867303
7	0.073	0.290893658160202	0.290893646138130	0.000000012022072
8	0.083	0.332180616975874	0.332180624328945	0.000000007353071
9	0.094	0.373360653390801	0.373360650613904	0.000000002776897
10	0.104	0.414420927792123	0.414420914948820	0.000000012843303
11	0.115	0.455348807851750	0.455348814214192	0.000000006362442
12	0.125	0.496131900643416	0.496131896972656	0.000000003670760
13	0.135	0.536758086765531	0.536758102158338	0.000000015392807
14	0.146	0.577215556157422	0.577215532753406	0.000000023404016
15	0.156	0.617492846061454	0.617492841556668	0.000000004504786
16	0.167	0.657578879934280	0.657578894238669	0.000000014304389
17	0.177	0.697463009513638	0.697462985588836	0.000000023924802
18	0.188	0.737135057991483	0.737135052680969	0.000000005310514
19	0.198	0.776585364395235	0.776585377594642	0.000000013199407
20	0.208	0.815804831293581	0.815804806916802	0.000000024376779
21	0.219	0.854784973465896	0.854784967377782	0.000000006088114
22	0.229	0.893517967715867	0.893517979800125	0.000000012084258
23	0.240	0.931996706198906	0.931996681429997	0.000000024768909
24	0.250	0.970214850588767	0.970214843750000	0.000000006838767
25	0.260	1.008166887247860	1.008166844211120	0.000000043036740
26	0.271	1.045848186680910	1.045848215224840	0.000000028543930
27	0.281	1.083255061847390	1.083255054429170	0.000000007418220
28	0.292	1.120384830850540	1.120384788034200	0.000000042816340
29	0.302	1.157235880118560	1.157235907305030	0.000000027186470
30	0.312	1.193807729112370	1.193807721138000	0.000000007974370
31	0.323	1.230101098369590	1.230101055799930	0.000000042569660
32	0.333	1.266117978191230	1.266118004039150	0.000000025847920
33	0.344	1.301861698875650	1.301861690357320	0.000000008518330
34	0.354	1.337337004272850	1.337336961931870	0.000000042340980
35	0.365	1.372550125986720	1.372550150535950	0.000000024549230
36	0.375	1.407508859161610	1.407508850097650	0.000000009063960
37	0.385	1.442222641507770	1.442222599324750	0.000000042183020
38	0.396	1.476702633031180	1.476702656345250	0.000000023314070
39	0.406	1.510961797403390	1.510961787775150	0.000000009628240
40	0.417	1.545014986412590	1.545014944254890	0.000000042157700

S. No.	Upper Limit	Value Calculated By Simpson's Three-Eighth Rule	Exact Value	Error in Calculated Value by using Simpson's three-eighth rule
41	0.427	1.578879025235070	1.578879047404070	0.000000022169000
42	0.438	1.612572799423680	1.612572789192200	0.000000010231480
43	0.448	1.646117344731990	1.646117302395160	0.000000042336830
44	0.458	1.679535937943690	1.679535959087210	0.000000021143520
45	0.469	1.712854189520090	1.712854178622360	0.000000010897730
46	0.479	1.746100138737280	1.746100095934380	0.000000042802900
47	0.490	1.779304350092210	1.779304370362420	0.000000020270210
48	0.500	1.812500011655000	1.812500000000000	0.000000011655000
49	0.510	1.845723035488450	1.845723086928520	0.000000051440070
50	0.521	1.879012159487680	1.879012083675480	0.000000075812200
51	0.531	1.912409051864960	1.912409039214250	0.000000012650710
52	0.542	1.945958416180120	1.945958467241490	0.000000051061370
53	0.552	1.979708099245960	1.979708021293880	0.000000077952080
54	0.562	2.013709201468550	2.013709187507620	0.000000013960930
55	0.573	2.048016186465710	2.048016237653010	0.000000051187300
56	0.583	2.082686995071190	2.082686913723510	0.000000081347680
57	0.594	2.117783161639270	2.117783145979040	0.000000015660230
58	0.604	2.153369928048740	2.153369979964680	0.000000051915940
59	0.615	2.189516363932060	2.189516277672100	0.000000086259960
60	0.625	2.226295489023340	2.226295471191400	0.000000017831940
61	0.635	2.263784391331400	2.263784444685740	0.000000053354340
62	0.646	2.302064353733350	2.302064260756280	0.000000092977070
63	0.656	2.341220982452270	2.341220961883660	0.000000020568610
64	0.667	2.381344329162000	2.381344384780800	0.000000055618800
65	0.677	2.422529024098220	2.422528922282450	0.000000101815770
66	0.688	2.464874410760020	2.464874386787410	0.000000023972610
67	0.698	2.508484671686890	2.508484730522150	0.000000058835260
68	0.708	2.553468968255010	2.553468855132210	0.000000113122800
69	0.719	2.599941581704620	2.599941553547970	0.000000028156650
70	0.729	2.648022042307000	2.648022105446540	0.000000063139540
71	0.740	2.697835276028550	2.697835148752270	0.000000127276280
72	0.750	2.749511751994340	2.749511718750000	0.000000033244340
73	0.760	2.803187614738800	2.803187683416420	0.000000068677620
74	0.771	2.859004837936020	2.859004693248820	0.000000144687200
75	0.781	2.917111378418360	2.917111339047550	0.000000039370810
76	0.792	2.977661311179820	2.977661386785770	0.000000075605950
77	0.802	3.040814990387270	3.040814824586530	0.000000165800740
78	0.812	3.106739210081960	3.106739163398740	0.000000046683220
79	0.823	3.175607341578700	3.175607425670410	0.000000084091710
80	0.833	3.247599501989710	3.247599310891910	0.000000191097800
81	0.844	3.322902721746220	3.322902666404840	0.000000055341380
82	0.854	3.401711084009690	3.401711178322760	0.000000094313070

S. No.	Upper Limit	Value Calculated By Simpson's Three-Eighth Rule	Exact Value	Error in Calculated Value by using Simpson's three-eighth rule
83	0.865	3.484225900957720	3.484225679861270	0.000000221096450
84	0.875	3.570655888272270	3.570655822753900	0.000000065518370
85	0.885	3.661217306151790	3.661217412611220	0.000000106459430
86	0.896	3.756134143626500	3.756133887273200	0.000000256353300
87	0.906	3.855638300170040	3.855638222768900	0.000000077401140
88	0.917	3.959969727872270	3.959969848603910	0.000000120731640
89	0.927	4.069376624070530	4.069376326605460	0.000000297465069
90	0.938	4.184115620251490	4.184115529060360	0.000000091191130
91	0.948	4.304451924914800	4.304452062257100	0.000000137342300
92	0.958	4.430659524826540	4.430659179756490	0.000000345070050
93	0.969	4.563021381388400	4.563021274283520	0.000000107104880
94	0.979	4.701829573692070	4.701829730207940	0.000000156515870
95	0.990	4.847385508720900	4.847385108871370	0.000000399849530
96	1.000	5.000000125374680	5.000000000000000	0.000000125374680

Graph-1: Error in the calculated values of integral-1 using Simpson's three-eighth rule at different values of upper limit



Conclusion

Highest values of errors in the calculated values of integrals using Simpson's three-eighth rule is given in Table-3. Minimum error has been observed in the case of integral-7 and maximum error in integral-4. Order of accuracy of the integrals is in the following sequence-

$$3 > 5 > 4 > 1 > 2$$

Highest value of error in each integral possesses at least six consecutive zeros after decimal point. Thus, we can say that the Simpson's three-eighth rule is accurate up to 6 places after the decimal point.

Table-3: Highest values of errors in the calculated values of integrals using Simpson's three-eighth rule

Integral No.	Lower Limit	Upper Limit	Length of Interval	Highest value of error in the calculated value using Simpson's three-eighth rule
1	0	1	1	0.000000399849530
2	1	3	2	0.000000833150400
3	0	1	1	0.000000055315960
4	-1	1	2	0.000000063049830
5	0	2	2	0.000000056440920

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