AN OVERVIEW ON EIGENVALUES AND EIGENVECTORS

Dr Shailendra Kumar Srivastava
Associate Professor
Department of Physics
MGPG College ,Gorakhpur

Dr Subhash Kumar Sharma*
Assistant Professor
Department Of Electronics
MGPG College ,Gorakhpur

Abstract

In this paper we will learn about the matrix eigenvalue problem AX = kX where A is a square matrix and k is a scalar (number). We will learn how to determine the eigenvalues (k) and corresponding eigenvectors (X) for a given matrix A. We will learn of some of the applications of eigenvalues and eigenvectors. Finally We will learn how eigenvalues and eigenvectors may be determined numerically.

Keywords: Eigen Values, Eigen Vectors, Determinant.

Basic Concepts:

From an applications viewpoint, eigenvalue problems are probably the most important problems that arise in connection with matrix analysis. In this Section we discuss the basic concepts. We shall see that eigenvalues and eigenvectors are associated with square matrices of order n × n. If n is small (2 or 3), determining eigenvalues is a fairly straightforward process (requiring the solutiuon of a low order polynomial equation). Obtaining eigenvectors is a little strange initially and it will help if you read this preliminary Section first.

Determinants : A square matrix possesses an associated determinant. Unlike a matrix, which is an array of numbers, a determinant has a single value

A two by two matrix $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ has an associated determinant

$$\det(C) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}\,c_{22} - c_{21}\,c_{12}$$

(Note square or round brackets denote a matrix, straight vertical lines denote a determinant.)

A three by three matrix has an associated determinant

$$\det(C) = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

Among other ways this determinant can be evaluated by an "expansion about the top row":

$$\det(C) = c_{11}\begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} - c_{12}\begin{vmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix} + c_{13}\begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix}$$

Note the minus sign in the second term.

A matrix such as

$$B = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$$

in the previous task which has zero determinant is called a singular matrix. The other two matrices A and C are non-singular. The key factor to be aware of is as follows:
Any non-singular n × n matrix C, for which det(C) ≠ 0, possesses an inverse C^{-1}
i.e. $CC^{-1} = C^{-1}C = I$ where I denotes the n × n identity matrix

A singular matrix does not possess an inverse.

Systems of linear equations

We first recall some basic results in linear (matrix) algebra. Consider a system of n equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{array}{ccccccccc} c_{11}x_1 & + & c_{12}x_2 & + & \ldots & + & c_{1n}x_n & = & k_1 \\ c_{21}x_1 & + & c_{22}x_2 & + & \ldots & + & c_{2n}x_n & = & k_2 \\ \vdots & + & \vdots & + & \ldots & + & \vdots & = & \vdots \\ c_{n1}x_1 & + & c_{n2}x_2 & + & \ldots & + & c_{nn}x_n & = & k_n \end{array}$$

We can write such a system in matrix form:

$$\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}, \qquad \text{or equivalently} \qquad CX = K.$$

We see that C is an $n \times n$ matrix (called the coefficient matrix), $X = \{x_1, x_2, \ldots, x_n\}^T$ is the $n \times 1$ column vector of unknowns and $K = \{k_1, k_2, \ldots, k_n\}^T$ is an $n \times 1$ column vector of given constants.

The zero matrix will be denoted by $\underline{O}$.

If $K \neq \underline{O}$ the system is called **inhomogeneous**; if $K = \underline{O}$ the system is called **homogeneous**.

Basic results in linear algebra

Consider the system of equations CX = K.

We are concerned with the nature of the solutions (if any) of this system. We shall see that this system only exhibits three solution types:

- The system is consistent and has a unique solution for X
- The system is consistent and has an infinite number of solutions for X
- The system is inconsistent and has no solution for X

There are two basic cases to consider:

$$\det(C) \neq 0 \quad \text{or} \quad \det(C) = 0$$

Case 1: $\det(C) \neq 0$

In this case C^{-1} exists and the **unique** solution to $CX = K$ is

$$X = C^{-1}K$$

Case 2: $\det(C) = 0$

In this case C^{-1} does not exist.

(a) If $K \neq \underline{O}$ the system $CA = K$ has **no solutions**.

(b) If $K = \underline{O}$ the system $CX = \underline{O}$ has an **infinite number of solutions.**

We note that a homogeneous system

$$CX = \underline{O}$$

has a unique solution $X = \underline{O}$ if $\det(C) \neq 0$ (this is called the **trivial solution**) or an infinite number of solutions if $\det(C) = 0$.

A simple eigenvalue problem:

We shall be interested in simultaneous equations of the form:
$AX = \lambda X$,
where A is an n × n matrix, X is an n × 1 column vector and λ is a scalar (a constant) and, in the first instance, we examine some simple examples to gain experience of solving problems of this type.

For Example:

Consider the following system with $n = 2$:

$$\begin{aligned} 2x + 3y &= \lambda x \\ 3x + 2y &= \lambda y \end{aligned}$$

so that

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

It appears that there are three unknowns x, y, λ. The obvious questions to ask are: can we find x, y? what is λ?

Solution

To solve this problem we firstly re-arrange the equations (take all unknowns onto one side);

$$(2-\lambda)x + 3y = 0 \qquad (1)$$

$$3x + (2-\lambda)y = 0 \qquad (2)$$

Therefore, from equation (2):

$$x = -\frac{(2-\lambda)}{3}y. \qquad (3)$$

Then when we substitute this into (1)

$$-\frac{(2-\lambda)^2}{3}y + 3y = 0 \quad \text{which simplifies to} \quad [-(2-\lambda)^2 + 9]\,y = 0.$$

We conclude that either $y = 0$ or $9 = (2-\lambda)^2$. There are thus two cases to consider:

Case 1

If $y = 0$ then $x = 0$ (from (3)) and we get the **trivial solution**. (We could have guessed this solution at the outset.)

Case 2

$$9 = (2-\lambda)^2$$

which gives, on taking square roots:

$$\pm 3 = 2 - \lambda \quad \text{giving} \quad \lambda = 2 \pm 3 \quad \text{so} \quad \lambda = 5 \quad \text{or} \quad \lambda = -1.$$

Now, from equation (3), if $\lambda = 5$ then $x = +y$ and if $\lambda = -1$ then $x = -y$.

We have now completed the analysis. We have found values for λ but we also see that we cannot obtain unique values for x and y: all we can find is the ratio between these quantities. This behaviour is typical, as we shall now see, of an eigenvalue problem.

2. General eigenvalue problems :

Consider a given square matrix A. If X is a column vector and λ is a scalar (a number) then the relation.

$$AX = \lambda X \qquad (4)$$

is called an eigenvalue problem. Our purpose is to carry out an analysis of this equation in a manner similar to the example above. However, we will attempt a more general approach which will apply to all problems of this kind.

Firstly, we can spot an obvious solution (for X) to these equations. The solution X = 0 is a possibility (for then both sides are zero). We will not be interested in these trivial solutions of the eigenvalue problem. Our main interest will be in the occurrence of non-trivial solutions for X. These may exist for special values of λ, called the eigenvalues of the matrix A. We proceed as in the previous example: take all unknowns to one side:

$$(A - \lambda I)X = 0 \qquad (5)$$

where I is a unit matrix with the same dimensions as A. (Note that AX − λX = 0 does not simplify to (A − λ)X = 0 as you cannot subtract a scalar λ from a matrix A). This equation (5) is a homogeneous system of equations. In the notation of the earlier discussion C ≡ A − λI and K ≡ 0. For such a system we know that non-trivial solutions will only exist if the determinant of the coefficient matrix is zero:

$$\det(A - \lambda I) = 0 \qquad (6)$$

Equation (6) is called the characteristic equation of the eigenvalue problem. We see that the characteristic equation only involves one unknown λ. The characteristic equation is generally a polynomial in λ, with degree being the same as the order of A (so if A is 2 × 2 the characteristic equation is a quadratic, if A is a 3 × 3 it is a cubic equation, and so on). For each value of λ that is obtained the corresponding value of X is obtained by solving the original equations (4). These X's are called eigenvectors.

N.B. We shall see that eigenvectors are only unique up to a multiplicative factor: i.e. if X satisfies AX = λX then so does kX when k is any constant.

Example

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

Solution

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{i.e.} \quad (A - \lambda I)X = 0.$$

Non-trivial solutions will exist if $\det(A - \lambda I) = 0$

that is, $\det\left\{\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 0, \quad \therefore \quad \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0,$

expanding this determinant: $(1-\lambda)(2-\lambda) = 0$. Hence the solutions for λ are: $\lambda = 1$ and $\lambda = 2$.

So we have found two values of λ for this 2×2 matrix A. Since these are unequal they are said to be **distinct** eigenvalues.

To each value of λ there corresponds an eigenvector. We now proceed to find the eigenvectors.

Case 1

$\lambda = 1$ (smaller eigenvalue). Then our original eigenvalue problem becomes: $AX = X$. In full this is

$$\begin{aligned} x &= x \\ x + 2y &= y \end{aligned}$$

Simplifying

$$\begin{aligned} x &= x & \text{(a)} \\ x + y &= 0 & \text{(b)} \end{aligned}$$

All we can deduce here is that $x = -y \quad \therefore \quad X = \begin{bmatrix} x \\ -x \end{bmatrix}$ for any $x \neq 0$

(We specify $x \neq 0$ as, otherwise, we would have the trivial solution.)

So the eigenvectors corresponding to eigenvalue $\lambda = 1$ are all proportional to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, e.g. $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ etc.

Sometimes we write the eigenvector in **normalised** form that is, with modulus or magnitude 1. Here, the normalised form of X is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{which **is unique**.}$$

Solution (contd.)

Case 2 Now we consider the larger eigenvalue $\lambda = 2$. Our original eigenvalue problem $AX = \lambda X$ becomes $AX = 2X$ which gives the following equations:

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.

$$\begin{aligned} x &= 2x \\ x + 2y &= 2y \end{aligned}$$

These equations imply that $x = 0$ whilst the variable y may take any value whatsoever (except zero as this gives the trivial solution).

Thus the eigenvector corresponding to eigenvalue $\lambda = 2$ has the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$, e.g. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ etc.

The normalised eigenvector here is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In conclusion: the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ has two eigenvalues and two associated normalised eigenvectors:

$$\lambda_1 = 1, \qquad \lambda_2 = 2$$

$$X_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For Example:

Find the eigenvalues and eigenvectors of the 3×3 matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution

The eigenvalues and eigenvectors are found by solving the eigenvalue problem

$$AX = \lambda X \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proceeding as in Example 5:

$(A - \lambda I)X = 0$ and non-trivial solutions for X will exist if $\det(A - \lambda I) = 0$

Solution (contd.)

that is,

$$\det\left\{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = 0$$

$$\text{i.e.} \quad \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

Expanding this determinant we find:

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

that is,

$$(2-\lambda)\{(2-\lambda)^2 - 1\} - (2-\lambda) = 0$$

Taking out the common factor $(2-\lambda)$:

$$(2-\lambda)\{4 - 4\lambda + \lambda^2 - 1 - 1\}$$

which gives: $(2-\lambda)[\lambda^2 - 4\lambda + 2] = 0.$

This is easily solved to give: $\lambda = 2$ or $\lambda = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}.$

So (typically) we have found three possible values of λ for this 3×3 matrix A.

To each value of λ there corresponds an eigenvector.

Case 1: $\lambda = 2 - \sqrt{2}$ (lowest eigenvalue)

Then $AX = (2-\sqrt{2})X$ implies

$$\begin{aligned} 2x - y &= (2-\sqrt{2})x \\ -x + 2y - z &= (2-\sqrt{2})y \\ -y + 2z &= (2-\sqrt{2})z \end{aligned}$$

Simplifying

$$\begin{aligned} \sqrt{2}x - y &= 0 & \text{(a)} \\ -x + \sqrt{2}y - z &= 0 & \text{(b)} \\ -y + \sqrt{2}z &= 0 & \text{(c)} \end{aligned}$$

We conclude the following:

(c) $\Rightarrow y = \sqrt{2}z$ (a) $\Rightarrow y = \sqrt{2}x$

$\therefore$ these two relations give $x = z$ then (b) $\Rightarrow -x + 2x - x = 0$

The last equation gives us no information; it simply states that $0 = 0$.

Solution (contd.)

$\therefore \quad X = \begin{bmatrix} x \\ \sqrt{2}x \\ x \end{bmatrix}$ for any $x \neq 0$ (otherwise we would have the trivial solution). So the eigenvectors corresponding to eigenvalue $\lambda = 2 - \sqrt{2}$ are all proportional to $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$.

In normalised form we have an eigenvector $\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$.

Case 2: $\lambda = 2$

Here $AX = 2X$ implies $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e.

$$\begin{aligned} 2x - y &= 2x \\ -x + 2y - z &= 2y \\ -y + 2z &= 2z \end{aligned}$$

After simplifying the equations become:

$$\begin{aligned} -y &= 0 & \text{(a)} \\ -x - z &= 0 & \text{(b)} \\ -y &= 0 & \text{(c)} \end{aligned}$$

(a), (c) imply $y = 0$; (b) implies $x = -z$

$\therefore$ eigenvector has the form $\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$ for any $x \neq 0$.

That is, eigenvectors corresponding to $\lambda = 2$ are all proportional to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

In normalised form we have an eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Solution (contd.)

Case 3: $\lambda = 2 + \sqrt{2}$ (largest eigenvalue)

Proceeding along similar lines to cases 1,2 above we find that the eigenvectors corresponding to $\lambda = 2 + \sqrt{2}$ are each proportional to $\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ with normalised eigenvector $\frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$.

In conclusion the matrix A has three distinct eigenvalues:

$$\lambda_1 = 2 - \sqrt{2}, \qquad \lambda_2 = 2 \qquad \lambda_3 = 2 + \sqrt{2}$$

and three corresponding normalised eigenvectors:

$$X_1 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \qquad X_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad X_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

3.Properties of eigenvalues and eigenvectors:

There are a number of general properties of eigenvalues and eigenvectors which you should be familiar with. You will be able to use them as a check on some of your calculations.

Property 1: Sum of eigenvalues

For any square matrix A: sum of eigenvalues = sum of diagonal terms of A (called the trace of A)

sum of eigenvalues = sum of diagonal terms of A (called the **trace** of A)

Formally, for an $n \times n$ matrix A: $\sum_{i=1}^{n} \lambda_i = \text{trace}(A)$

(Repeated eigenvalues must be counted according to their multiplicity.)

Thus if $\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = 1$ then $\sum_{i=1}^{3} \lambda_i = 9$).

Property 2: Product of eigenvalues

For any square matrix A:

product of eigenvalues = determinant of A

Formally: $\lambda_1\lambda_2\lambda_3 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i = \det(A)$

The symbol $\prod$ simply denotes multiplication, as $\sum$ denotes summation.

Property 3: Linear independence of eigenvectors:

Eigenvectors of a matrix A corresponding to distinct eigenvalues are linearly independent i.e. one eigenvector cannot be written as a linear sum of the other eigenvectors. The proof of this result is omitted but we illustrate this property with two examples.

We saw earlier that the Matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

has distinct eigenvalues $\lambda_1 = 1$ $\lambda_2 = 2$ with associated eigenvectors

$$X^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad X^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

respectively.

Clearly $X^{(1)}$ is **not** a constant multiple of $X^{(2)}$ and these eigenvectors are **linearly independent**.

We also saw that the 3×3 matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

had the following distinct eigenvalues $\lambda_1 = 2 - \sqrt{2}$, $\lambda_2 = 2$, $\lambda_3 = 2 + \sqrt{2}$ with corresponding eigenvectors of the form shown:

$$X^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \qquad X^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad X^{(3)} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

Clearly none of these eigenvectors is a constant multiple of any other. Nor is any one obtainable as a linear combination of the other two. The three eigenvectors are linearly independent.

Property 4: Eigenvalues of diagonal matrices

A 2×2 diagonal matrix D has the form

$$D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

The characteristic equation

$$|D - \lambda I| = 0 \quad \text{is} \quad \begin{vmatrix} a - \lambda & 0 \\ 0 & d - \lambda \end{vmatrix} = 0$$

i.e. $(a - \lambda)(d - \lambda) = 0$

So the eigenvalues are simply the diagonal elements a and d.

Similarly a 3×3 diagonal matrix has the form

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

having characteristic equation

$|D - \lambda I| = (a - \lambda)(b - \lambda)(c - \lambda) = 0$

so again the diagonal elements are the eigenvalues.

We can see that a diagonal matrix is a particularly simple matrix to work with. In addition to the eigenvalues being obtainable immediately by inspection it is exceptionally easy to multiply diagonal matrices.

Conclusions:

This paper discusses the conditioning of eigenvalues of matrices. The simple structure of these matrices makes it possible to derive simple expressions and bounds for the individual, global, traditional, and structured condition numbers. This led us to discuss several applications, including an inverse eigenvalue problem. These applications are very promising and will be investigated in more detail in forthcoming work.

References:

[1] D. J. Aldous and J. Fill: Reversible Markov Chains and Random Walks on Graphs (book in preparation); URL for draft at http://www.stat.Berkeley.edu/users/aldous

[2] F. Alizadeh: Combinatorial optimization with semi-definite matrices, in: Integer Programming and Combinatorial Optimization (Proceedings of IPCO '92), (eds. E. Balas, G. Cornu´ejols and R. Kannan), Carnegie Mellon University Printing (1992), 385–405.

[3] F. Alizadeh, Interior point methods in semidefinite programming with applications to combinatorial optimization, SIAM J. Optim. 5 (1995), 13–51.

[4] F. Alizadeh, J.-P. Haeberly, and M. Overton: Complementarity and nondegeneracy in semidefinite programming, in: Semidefinite Programming, Math. Programming Ser. B, 77 (1997), 111–128.

[5] N. Alon: Eigenvalues and expanders, Combinatorica 6(1986), 83–96.

[6] N. Alon, The Shannon capacity of a union, Combinatorica 18 (1998), 301–310.

[7] N. Alon and N. Kahale: Approximating the independence number via the ϑ-function, Math. Programming 80 (1998), Ser. A, 253–264.

[8] N. Alon and V. D. Milman: λ1, isoperimetric inequalities for graphs and superconcentrators, J. Combinatorial Theory B 38(1985), 73–88.

[9] N. Alon, L. Lov´asz: Unextendible product bases, J. Combin. Theory Ser. A 95 (2001), 169–179.

[10] N. Alon, P.D. Seymour and R. Thomas: Planar separators, SIAM J. Disc. Math. 7 (1994), 184–193.

[11] N. Alon and J.H. Spencer: The Probabilistic Method, Wiley, New York, 1992.

[12] E. Andre'ev: On convex polyhedra in Lobachevsky spaces, Mat. Sbornik, Nov. Ser. 81 (1970), 445–478.

[13] S. Arora, C. Lund, R. Motwani, M. Sudan, M. Szegedy: Proof verification and hardness of approximation problems Proc. 33rd FOCS (1992), 14–23.