Quadruple of Self Maps in Metric Space and Common Fixed Point Theorems

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***Abstract*: Using property (*E.A.*) and its variants, we prove some common fixed point theorems for quadruple of weakly compatible self maps in metric space in this paper. Our results extend and unify various known results in literature. We also give application of proved result for four finite families of maps.**

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1. Introduction

In 2011, Rao and Pant [4] utilized the concept of finite metric spaces and proved some common fixed point theorems for asymptotically regular maps. Recently, Mishra et. al. [4] proved some common fixed point theorems using property (*E.A.*) (which was introduced by Aamri and Moutawakil [1]) in metric spaces.

Using property (*E.A.*) and its variants, we prove some common fixed point theorems for quadruple of weakly compatible self maps in metric space in this paper. Our results extend and unify various known results in literature. We also give application of proved result for four finite families of maps.

Our results extend and unify various known results in literature such as Ghilzean[5], Rao and Pant [4], Mishra et. al.[7] and Rudeanu[8].

2. Preliminaries

Definition 2.1 [6]. Two self maps A and S of a metric space are weakly compatible if $ASx = SAx$ for all x at which $Ax = Sx$.

Definition 2.2 [1]. Self maps A and S on a satisfies the property (*E.A)* if there exist a sequence $\{x_n\}$ in V such that $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = z$ for some $z \in V$.

Clearly, both compatible and noncompatible pairs enjoy property (*E.A*).

Definition 2.3 [2]. Two pairs of self maps (A, S) and (B, T) on a satisfy common property (*E.A)* if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in V such that $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} B(y_n) = p$ for some $p \in V$.

Definition 2.4 [3]. Two pairs of self maps (A, S) and (B, T) on a satisfy the ($JCLR_{ST}$) property (with respect to mappings S and T) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in V such that

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} B(y_n) = Sz = Tz \text{ where } z \in V.$$

Definition 2.5 [2]. Two finite families of self maps $\{A_i\}_{i=1}^{m}$ and $\{B_j\}_{j=1}^{n}$ on a set X are pairwise commuting if

(i) $A_i A_j = A_j A_i$, $i, j \in \{1,2,3,...m\}$,

(ii) $B_i B_j = B_j B_i$, $i, j \in \{1,2,3,...n\}$,

(iii) $A_i B_j = B_j A_i$, $i \in \{1,2,3,...m\}$, $j \in \{1,2,3,...n\}$.

3. Main Results

Let Φ be the set of all continuous functions $\Psi : X \to X$ satisfying $\Psi(a) < a$ for all a in X.

Theorem 3.1: Let A, B, S and T be four self maps in metric space (X, d) satisfying:

(3.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;

(3.2) there exist $\Psi \in \Phi$ such that

$$d(Ax, By) = \Psi(M(x, y)) \text{ where } M(x, y) = \max\{d(Sx, Ty), d(Sx, Ax), d(By, Ty)\}$$

for all $x, y \in X$;

(3.3) pair (A,S) or (B,T) satisfies the property ($E.A$).

(3.4) range of one of the maps *A*, *B*, *S* or *T* is a closed subspace of *X*.
Then pairs (*A*, *S*) and (*B*, *T*) have coincidence point. Further if (*A*, *S*) and (*B*, *T*) be weakly compatible pairs of self maps of *X* then *A*, *B*, *S* and *T* have a unique common fixed point in X.

Proof: If the pair (*B*, *T*) satisfies the property (*E.A.*), then there exist a sequence $\{x_n\}$ in *X* such that $Bx_n \to z$ and $Tx_n \to z$ for some $z \in X$ as $n \to \infty$.

Since, $B(X) \subset S(X)$, therefore, there exist a sequence $\{y_n\}$ in *V* such that $Bx_n = Sy_n$. Hence, $Sy_n \to z$ as $n \to \infty$. Also, since $A(X) \subset T(X)$, there exist a sequence $\{z_n\}$ in *X* such that $Tx_n = Az_n$. Hence, $Az_n \to z$ as $n \to \infty$.

Suppose that $S(X)$ is a closed subspace of *X*. Then $z = Su$ for some $u \in X$. Therefore, $Az_n \to Su, Bx_n \to Su, Tx_n \to Su, Sy_n \to Su$ as $n \to \infty$.

First we claim that $Au = Su$. Suppose not, then by (3.2), take $x = u$, $y = x_n$, we get

$$d(Au, Bx_n) = \Psi(M(u, x_n))$$

As $n \to \infty$

$$d(Au, Su) = \Psi\left(\lim_{n\to\infty} M(u, x_n)\right) \text{.....(3.5)}$$

where

$$M(u, x_n) = \max\{d(Su, Tx_n), d(Su, Au), d(Bx_n, Tx_n)\}$$

As $n \to \infty$

$$\lim_{n\to\infty} M(u, x_n) = \max\{d(Su, Su), d(Su, Au), d(Su, Su)\}$$
$$= d(Su, Au)$$

(3.5) gives,

$$d(Au, Su) = \Psi(d(Au, Su)) < (d(Au, Su))$$

a contradiction, hence, $Au = Su$. As *A* and *S* are weakly compatible. Therefore,
$ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $A(X) \subseteq T(X)$, there exist $v \in X$ such that $Au = Tv$. We now show that, $Tv = Bv$. Suppose not, then by (3.2), take $x = u$, $y = v$, we have,

$$d(Au, Bv) = \Psi(M(u, v))$$

or

$$d(Tv, Bv) = \Psi(M(u, v)) \text{....(3.6)}$$

where

$$M(u, v) = \max\{d(Su, Tv), d(Su, Au), d(Bv, Tv)\}$$
$$= \max\{d(Tv, Tv), d(Au, Au), d(Bv, Tv)\}$$
$$= d(Bv, Tv)$$

Thus, (3.6) gives,

$$d(Tv, Bv) = \Psi(d(Bv, Tv)) < (d(Bv, Tv))$$

a contradiction, hence, $Bv = Tv$.
As *B* and *T* are weakly compatible, therefore, $BTv = TBv$ and hence, $BTv = TBv = TTv = BBv$.
Next we claim that $AAu = Au$.
Suppose not, then by (3.2), take $x = Au$, $y = v$, we get

$$d(AAu, Bv) = \Psi(M(Au, v)) \text{....(3.7)}$$

where

$$M(Au, v) = \max\{d(SAu, Tv), d(SAu, AAu), d(Bv, Tv)\}$$
$$= \max\{d(AAu, Bv), d(AAu, AAu), d(Tv, Tv)\}$$
$$= d(AAu, Bv)$$

(3.7) gives,

$$d(AAu, Bv) = \Psi(d(AAu, Bv)) < d(AAu, Bv)$$

again a contradiction, hence $AAu = Au$. Therefore, $Au = AAu = SAu$ and Au is a common fixed point of *A* and *S*. Similarly, we can prove that Bv is a common fixed point of *B* and *T*. As $Au = Bv$, we conclude that Au is a common fixed point of *A*, *B*, *S* and *T*.

The proof is similar when $T(X)$ is assumed to be a closed subspace of X. The cases in which $A(X)$ or $B(X)$ is a closed subspace of X are similar to the cases in which $T(X)$ or $S(X)$ respectively, is closed since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

For uniqueness, let u and v are two common fixed points of A, B, S and T. Therefore, by definition, $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$. Then by (3.2), take $x = u$ and $y = v$, we get

$$d(Au, Bv) = \Psi\left(M(u,v)\right)$$

or

$$d(u,v) = \Psi\left(M(u,v)\right) \ldots\ldots(3.8)$$

where

$$\begin{aligned} M(u,v) &= \max\left\{d(Su,Tv), d(Su,Au), d(Bv,Tv)\right\} \\ &= \max\left\{d(u,v), d(u,u), d(v,v)\right\} \\ &= d(u,v) \end{aligned}$$

Equation (3.8) gives,

$$d(u,v) = \Psi\left(d(u,v)\right) < d(u,v)$$

a contradiction, therefore, $u = v$. Hence A, B, S and T have a unique common fixed point in X.

Taking $B = A$ and $T = S$ in Theorem 3.1, we get following result:

Corollary 3.1: Let A and S be two self maps in metric space (X,d) such that

(3.9) there exist $\Psi \in \Phi$ and $x, y \in X$ such that

$$d(Ax, Ay) = \Psi\left(M(x,y)\right) \text{ where}$$

$$M(x,y) = \max\left\{d(Sx,Sy), d(Sx,Ax), d(Ay,Sy)\right\};$$

(3.10) pair (A,S) satisfies the property $(E.A)$

(3.11) the range of one of the maps A *or* S is a closed subspace of X

Then A and S have a coincidence point in X. Further if (A, S) be weakly compatible pair of self maps then A and S have a unique common fixed point in X.

As an application of Theorem 3.1, we prove a common fixed point theorem for four finite families of maps. While proving our result, we utilize Definition 2.9 which is a natural extension of commutativity condition to two finite families.

Theorem 3.4: Let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_n\}$, $\{S_1, S_2, \ldots, S_p\}$ and $\{T_1, T_2, \ldots, T_q\}$ be four finite families of self maps of a metric space (X,d) such that $A = A_1.A_2.\ldots.A_m$, $B = B_1.B_2.\ldots.B_n$, $S = S_1.S_2.\ldots.S_p$ and $T = T_1.T_2.\ldots.T_q$ satisfy the condition (3.2) and

(3.21) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$)

(3.22) the pair (A,S) (or (B,T)) satisfy property $(E.A)$.

Then the pairs (A,S) and (B,T) have a point of coincidence each. Moreover finite families of self maps A_i, S_k, B_r and T_t have a unique common fixed point provided that the pairs of families $(\{A_i\},\{S_k\})$ and $(\{B_r\},\{T_t\})$ commute pairwise for all $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, p$, $r = 1, 2, \ldots, n$, $t = 1, 2, \ldots, q$.

Proof: Since self maps A, B, S, T satisfy all the conditions of theorem 3.1, the pairs (A,S) and (B,T) have a point of coincidence. Also the pairs of families $(\{A_i\},\{S_k\})$ and $(\{B_r\},\{T_t\})$ commute pairwise, we first show that $AS = SA$ as

$$\begin{aligned} AS &= (A_1A_2 \ldots A_m)(S_1S_2 \ldots S_p) = (A_1A_2 \ldots A_{m-1})(A_m S_1S_2 \ldots S_p) \\ &= (A_1A_2 \ldots A_{m-1})(S_1S_2 \ldots S_p A_m) = (A_1A_2 \ldots A_{m-2})(A_{m-1} S_1S_2 \ldots S_p A_m) \\ &= (A_1A_2 \ldots A_{m-2})(S_1S_2 \ldots S_p A_{m-1}A_m) = \ldots = A_1(S_1S_2 \ldots S_p A_2 \ldots A_m) \\ &= (S_1S_2 \ldots S_p)(A_1A_2 \ldots A_m) = SA. \end{aligned}$$

Similarly one can prove that $BT = TB$. And hence, obviously the pair (A, S) and (B, T) are weakly compatible. Now using Theorem 3.1, we conclude that A, S, B and T have a unique common fixed point in V, say z.

Now, one needs to prove that z remains the fixed point of all the component maps.

For this consider

$$\begin{aligned} A(A_i z) &= ((A_1A_2 \ldots A_m)A_i)z = (A_1A_2 \ldots A_{m-1})(A_m A_i)z \\ &= (A_1A_2 \ldots A_{m-1})(A_i A_m)z = (A_1A_2 \ldots A_{m-2})(A_{m-1} A_i A_m)z \\ &= (A_1A_2 \ldots A_{m-2})(A_i A_{m-1} A_m)z = \ldots = A_1(A_iA_2 \ldots A_m)z \\ &= (A_1A_i)(A_2 \ldots A_m)z \end{aligned}$$

$= (A_i A_1)(A_2 \ldots A_m)z = A_i (A_1 A_2 \ldots A_m)z = Ai\, Az = A_i z.$
Similarly, one can prove that
$A(S_k z) = S_k (Az) = S_k z$, $S(S_k z) = S_k(Sz) = S_k z$,
$S(A_i z) = A_i(Sz) = A_i z$, $B(B_r z) = B_r(Bz) = B_r z$,
$B(T_t z) = T_t(Bz) = T_t z$, $T(T_t z) = T_t(Tz) = T_t z$
and
$T(B_r z) = B_r(Tz) = B_r z$,
which shows that (for all i, r, k and t) $A_i z$ and $S_k z$ are other fixed point of the pair (A, S) whereas $B_r z$ and $T_t z$ are other fixed points of the pair (B, T). As A, B, S and T have a unique common fixed point, so, we get
$z = A_i z = S_k z = B_r z = T_t z$, for all $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, p$,
$r = 1, 2, \ldots, n$, $t = 1, 2, \ldots, q$.
which shows that z is a unique common fixed point of $\{A_i\}_{i=1}^{m}$, $\{S_k\}_{k=1}^{p}$, $\{B_r\}_{r=1}^{n}$ and $\{T_t\}_{t=1}^{q}$.

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